ON J-SELFADJOINT EXTENSIONS OF J-SYMMETRIC OPERATORS

IAN KNOWLES

Abstract. A short proof is given (via the theory of conjugate-linear operators) of the fact that every J-symmetric operator in a Hilbert space $\mathcal{H}$ has a J-selfadjoint extension in $\mathcal{H}$.

An operator $J$ on a complex Hilbert space $\mathcal{H}$ is called a conjugation if it is an involution and

$$\langle Jx, Jy \rangle = \langle y, x \rangle \quad (1)$$

for all $x$ and $y$ in $\mathcal{H}$. Following I. M. Glazman [3], a linear operator $A$ in $\mathcal{H}$ with domain $\mathcal{D}(A)$ dense in $\mathcal{H}$ is called $J$-symmetric if

$$\langle Ax, Jy \rangle = \langle x, JAy \rangle \quad (2)$$

for all $x$ and $y$ in $\mathcal{D}(A)$. Clearly, $A$ is $J$-symmetric if and only if $JAJ \subset A^*$. If $JAJ = A^*$, then $A$ is said to be $J$-selfadjoint.

Now, it is well known that symmetric linear operators in $\mathcal{H}$ need not possess selfadjoint extensions; indeed, it is a basic fact in the theory that these operators are precisely those with unequal deficiency indices (see [1, Corollary 13, p. 1230]). It is thus rather surprising to learn that the following result is known for $J$-symmetric operators.

Theorem A [2]. Every $J$-symmetric operator in $\mathcal{H}$ has a $J$-selfadjoint extension in $\mathcal{H}$.

The method of proof used in [2] makes use of certain properties of the graphs of the operators concerned. We return to this method later. Our main objective here is to give a simplified proof of this result.

The method that we use is based upon a consideration of the analogous theory for conjugate-linear operators in $\mathcal{H}$. Here, an operator $T$, with domain $\mathcal{D}(T)$ dense in $\mathcal{H}$, is said to be conjugate-linear if

$$T(ax + \beta y) = \bar{a}Tx + \beta Ty$$

for all scalars $a$ and $\beta$, and all $x$ and $y$ in $\mathcal{D}(T)$. By analogy with the corresponding theory for linear operators, we define $\mathcal{D}(T^*)$ to be the set of elements $y$ in $\mathcal{H}$ for which there corresponds an element $z$ in $\mathcal{H}$ (necessarily unique) such that $\langle Tx, y \rangle = \langle z, x \rangle$ holds for all $x$ in $\mathcal{D}(T)$. The adjoint, $T^*$, of $T$ is defined on $\mathcal{D}(T^*)$ by the equation $T^*y = z$. Thus
\[(Tx, y) = (T^* y, x)\]  \hspace{1cm} (3)

for all \(x \in \mathcal{D}(T)\) and \(y \in \mathcal{D}(T^*)\). The conjugate-linear operator \(T\) is said to be symmetric if

\[(Tx, y) = (Ty, x)\]  \hspace{1cm} (4)

for all \(x\) and \(y\) in \(\mathcal{D}(T)\); thus \(T\) is symmetric if and only if \(T \subset T^*\). If \(T = T^*\), then \(T\) is called selfadjoint. Finally, we say that \(T\) is maximal symmetric if \(T\) is symmetric and possesses no proper symmetric extensions. The usual maximality argument, using Zorn’s lemma, shows that every symmetric conjugate-linear operator has a maximal symmetric extension. Thus, if we observe that the linear operator \(A\) is \(J\)-symmetric (\(J\)-selfadjoint) if and only if the conjugate-linear operator \(JA\) is symmetric (selfadjoint), then Theorem A may be equivalently formulated as,

**Theorem B.** Every maximal symmetric conjugate-linear operator is selfadjoint.

**Proof.** If we assume to the contrary, that the theorem is not true, then there is a conjugate-linear operator \(T\) that is maximal symmetric, but not selfadjoint.

Let \(y \in \mathcal{D}(T^*) \setminus \mathcal{D}(T)\), and define an extension \(T_1\) of \(T\) by

\[\mathcal{D}(T_1) = \mathcal{D}(T) + \{\alpha y\}, \quad \alpha \in \mathbb{C},\]

and, for \(x = p + q \in \mathcal{D}(T_1)\) where \(p \in \mathcal{D}(T)\) and \(q \in \{\alpha y\}\) set

\[T_1x = Tp + T^*q.\]

Then, for \(x = p_1 + q_1\) and \(y = p_2 + q_2\) in \(\mathcal{D}(T_1)\), where \(q_1 = \alpha_1 y\) and \(q_2 = \alpha_2 y\) for some complex scalars \(\alpha_1\) and \(\alpha_2\), we have

\[
(T_1x, y) = (Tp_1, p_2) + (Tp_1, q_2) + (T^*q_1, p_2) + (T^*q_1, q_2) \\
= (Tp_2, p_1) + (T^*q_2, p_1) + (Tp_2, q_1) + \alpha_1 \alpha_2 (T^*y, y)
\]

by (3) and (4)

\[= (T_1y, x).\]

Thus \(T_1\) is a proper symmetric extension of \(T\), in contradiction to the maximality of \(T\). The proof is complete. \(\square\)

As a final result we have:

**Theorem C.** Let \(\mathcal{K}_2 = \mathcal{K} \oplus \mathcal{K}\), and let \(G_A = \{(x, y) \in \mathcal{K}_2 : y = Ax\}\) denote the graph of the closed \(J\)-symmetric operator \(A\). Let \(T : \mathcal{K}_2 \to \mathcal{K}_2\) be defined by \(T[x, y] = [Jy, -Jx]\), and set \(D = \mathcal{K}_2 \ominus [G_A \oplus TG_A]\). Then

(i) there exists a subspace \(X\) in \(\mathcal{K}_2\) such that \(X \oplus TX = \mathcal{D}\), and the set \(G_A \oplus X\) is the graph of a \(J\)-selfadjoint extension of \(A\),

(ii) every \(J\)-selfadjoint extension of \(A\) has the form given in (i).

**Proof.** Part (i) is precisely the result proved in [2]. Alternatively, part (i) may be derived from Theorem A, and part (ii), which we now prove. Let \(B\) be an arbitrary \(J\)-selfadjoint extension of \(A\), with graph \(G_B\), and set \(X = G_B \ominus G_A\). Since \(G_{JB^*J} = TG_B^\perp\), and \(B\) is \(J\)-selfadjoint, it is clear that \(G_B \oplus TG_B = \mathcal{K}_2\), and thus that

\[X \oplus G_A \oplus T(X \oplus G_A) = \mathcal{K}_2.\]  \hspace{1cm} (5)
Finally, as $X \perp G_{A}$, it is not hard to see that $TX \perp TG_{A}$. The result now follows from (5). □

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CALIFORNIA 94720

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG, SOUTH AFRICA