ON J-SELFADJOINT EXTENSIONS OF J-SYMMETRIC OPERATORS

IAN KNOWLES

Abstract. A short proof is given (via the theory of conjugate-linear operators) of the fact that every J-symmetric operator in a Hilbert space \( \mathcal{H} \) has a J-selfadjoint extension in \( \mathcal{H} \).

An operator \( J \) on a complex Hilbert space \( \mathcal{H} \) is called a conjugation if it is an involution and
\[
(Jx, Jy) = (y, x)
\]
for all \( x \) and \( y \) in \( \mathcal{H} \). Following I. M. Glazman [3], a linear operator \( A \) in \( \mathcal{H} \) with domain \( \mathcal{D}(A) \) dense in \( \mathcal{H} \) is called J-symmetric if
\[
(Ax, Jy) = (x, JAy)
\]
for all \( x \) and \( y \) in \( \mathcal{D}(A) \). Clearly, \( A \) is J-symmetric if and only if \( JAJ \subseteq A^* \). If \( JAJ = A^* \), then \( A \) is said to be J-selfadjoint.

Now, it is well known that symmetric linear operators in \( \mathcal{H} \) need not possess selfadjoint extensions; indeed, it is a basic fact in the theory that these operators are precisely those with unequal deficiency indices (see [1, Corollary 13, p. 1230]). It is thus rather surprising to learn that the following result is known for J-symmetric operators.

Theorem A [2]. Every J-symmetric operator in \( \mathcal{H} \) has a J-selfadjoint extension in \( \mathcal{H} \).

The method of proof used in [2] makes use of certain properties of the graphs of the operators concerned. We return to this method later. Our main objective here is to give a simplified proof of this result.

The method that we use is based upon a consideration of the analogous theory for conjugate-linear operators in \( \mathcal{H} \). Here, an operator \( T \), with domain \( \mathcal{D}(T) \) dense in \( \mathcal{H} \), is said to be conjugate-linear if
\[
T(ax + \beta y) = \alpha Tx + \beta Ty
\]
for all scalars \( \alpha \) and \( \beta \), and all \( x \) and \( y \) in \( \mathcal{D}(T) \). By analogy with the corresponding theory for linear operators, we define \( \mathcal{D}(T^*) \) to be the set of elements \( y \) in \( \mathcal{H} \) for which there corresponds an element \( z \) in \( \mathcal{H} \) (necessarily unique) such that \( (Tx, y) = (z, x) \) holds for all \( x \) in \( \mathcal{D}(T) \). The adjoint, \( T^* \), of \( T \) is defined on \( \mathcal{D}(T^*) \) by the equation \( T^*y = z \). Thus
for all $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$. The conjugate-linear operator $T$ is said to be symmetric if

$$ (Tx, y) = (Ty, x) $$

for all $x$ and $y$ in $\mathcal{D}(T)$; thus $T$ is symmetric if and only if $T \subseteq T^*$. If $T = T^*$, then $T$ is called selfadjoint. Finally, we say that $T$ is maximal symmetric if $T$ is symmetric and possesses no proper symmetric extensions. The usual maximality argument, using Zorn's lemma, shows that every symmetric conjugate-linear operator has a maximal symmetric extension. Thus, if we observe that the linear operator $A$ is $J$-symmetric ($J$-selfadjoint) if and only if the conjugate-linear operator $JA$ is symmetric (selfadjoint), then Theorem A may be equivalently formulated as,

**Theorem B.** Every maximal symmetric conjugate-linear operator is selfadjoint.

**Proof.** If we assume to the contrary, that the theorem is not true, then there is a conjugate-linear operator $T$ that is maximal symmetric, but not selfadjoint.

Let $y \in \mathcal{D}(T^*) \setminus \mathcal{D}(T)$, and define an extension $T_1$ of $T$ by

$$ \mathcal{D}(T_1) = \mathcal{D}(T) + \{ \alpha y \}, \quad \alpha \in \mathbb{C}, $$

and, for $x = p + q \in \mathcal{D}(T_1)$ where $p \in \mathcal{D}(T)$ and $q \in \{ \alpha y \}$ set

$$ T_1x = Tp + T^*q. $$

Then, for $x = p_1 + q_1$ and $y = p_2 + q_2$ in $\mathcal{D}(T_1)$, where $q_1 = \alpha_1 y$ and $q_2 = \alpha_2 y$ for some complex scalars $\alpha_1$ and $\alpha_2$, we have

$$ (T_1x, y) = (Tp_1, p_2) + (Tp_1, q_2) + (T^*q_1, p_2) + (T^*q_1, q_2) $$

$$ = (Tp_2, p_1) + (T^*q_2, p_1) + (Tp_2, q_1) + \bar{\alpha}_1 \bar{\alpha}_2 \langle T^*y, y \rangle $$

by (3) and (4)

$$ = (T_1y, x). $$

Thus $T_1$ is a proper symmetric extension of $T$, in contradiction to the maximality of $T$. The proof is complete. □

As a final result we have:

**Theorem C.** Let $\mathcal{K}_2 = \mathcal{K} \oplus \mathcal{K}$, and let $G_A = \{ [x, y] \in \mathcal{K}_2 : y = Ax \}$ denote the graph of the closed $J$-symmetric operator $A$. Let $T: \mathcal{K}_2 \rightarrow \mathcal{K}_2$ be defined by $T[x, y] = [Jy, -Jx]$, and set $D = \mathcal{K}_2 \ominus [G_A \oplus TG_A]$. Then

(i) there exists a subspace $X$ in $\mathcal{K}_2$ such that $X \oplus TX = D$, and the set $G_A \oplus X$ is the graph of a $J$-selfadjoint extension of $A$,

(ii) every $J$-selfadjoint extension of $A$ has the form given in (i).

**Proof.** Part (i) is precisely the result proved in [2]. Alternatively, part (i) may be derived from Theorem A, and part (ii), which we now prove. Let $B$ be an arbitrary $J$-selfadjoint extension of $A$, with graph $G_B$, and set $X = G_B \ominus G_A$. Since $G_{JB^*J} = TG_B^*$, and $B$ is $J$-selfadjoint, it is clear that $G_B \oplus TG_B = \mathcal{K}_2$, and thus that

$$ X \oplus G_A \oplus T(X \oplus G_A) = \mathcal{K}_2. $$

(5)
Finally, as $X \perp G_A$, it is not hard to see that $TX \perp TG_A$. The result now follows from (5). \qed

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT BERKELEY, BERKELEY, CALIFORNIA 94720

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG, SOUTH AFRICA