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AN INVARIANT SUBSPACE LATTICE OF ORDER-TYPE $\omega + \omega + 1$

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ABSTRACT. An example is given of a bounded linear operator on a Hilbert space whose lattice of invariant subspaces is totally ordered and is of order-type $\omega + \omega + 1$.

1. Introduction. If A is a (bounded) operator on a (complex) Hilbert space $\mathcal{H}$, then Lat A, the set of all (closed) subspaces of $\mathcal{H}$ invariant under A, is a complete lattice under the partial order defined by inclusion.

We say an abstract lattice l is *attainable* if there is an operator A on a separable, infinite-dimensional Hilbert space such that Lat A is order-isomorphic to l. Since Lat A is in general very hard to determine, few lattices are known to be attainable. The still unsolved invariant subspace problem asks whether the lattice 2 is attainable.

It is easy to determine the invariant subspace lattice for certain normal operators, but for nonnormal operators it is usually much harder. Nonnormal operators whose invariant subspace lattices are known include the unilateral shift ([**1**] and [**6**]), the Volterra operator ([**3**], [**7**], [**8**] and [**11**]), and certain weighted shifts ([**3**], [**4**], [**5**] and [**9**]). Other attainable lattices have been obtained by putting together known examples ([**2**] and [**10**]). For example, the direct product of countably many attainable lattices is attainable [**2**].

In this paper it is shown that $\omega + \omega + 1$ is attainable (or more precisely each chain of order-type $\omega + \omega + 1$ is attainable; chains are totally ordered lattices, and $\omega + \omega + 1$ is the order-type of the set $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \beta_1, \beta_2, \beta_3, \ldots, \gamma\}$, where the order increases from left to right).

2. The operator. The question: "Is $\omega + \omega + 1$ attainable?" was first raised by P. R. Halmos. It is mentioned in [**10**], and the operator whose Lat will be shown to be isomorphic to $\omega + \omega + 1$ is of the type suggested there as a solution. We now proceed to define the operator.

Let $\{\phi_0, \phi_1, \phi_2, \ldots\}$ be an orthonormal basis of a Hilbert space $\mathcal{H}$, and let A and B denote the backward weighted shifts on $\mathcal{H}$ determined by the equations

$$A\phi_n = \exp[1 - 2n]\phi_{n-1}, \qquad n = 1, 2, 3, \ldots, \quad \text{and} \quad A\phi_0 = 0,$$
$$B\phi_n = \exp[2 - 4n]\phi_{n-1}, \qquad n = 1, 2, 3, \ldots, \quad \text{and} \quad B\phi_0 = 0. \tag{2.1}$$

Let $\mu(1), \mu(2), \mu(3), \ldots$ be any increasing sequence of positive integers which

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satisfy the inequalities

$$\mu(n+1) > 5\mu(n), \qquad n = 1, 2, 3, \ldots, \tag{2.2}$$

and let C denote the rank one operator $\phi_0 \otimes \psi$, where $\psi = \sum_{s=1}^{\infty} \exp[-s]\phi_{\mu(s)}$. Thus $Cf = \langle f, \phi_0 \rangle \psi$ for each vector f in $\mathcal{H}$, where $\langle\, , \rangle$ is the inner product on $\mathcal{H}$, and in particular

$$C\phi_0 = \psi \quad \text{and} \quad C\phi_n = 0, \qquad n = 1, 2, 3, \ldots. \tag{2.3}$$

Let $\mathcal{H}$ denote the orthogonal sum of two copies of $\mathcal{H}$ and let T denote the operator on $\mathcal{H}$ whose matrix representation is $\left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]$. Thus for each vector $f \oplus g$ in $\mathcal{H}$, $T(f \oplus g) = (Af + Cg) \oplus Bg$. We shall show that Lat T is isomorphic to $\omega + \omega + 1$.

The following simple proposition is stated without proof. It will be used to calculate the norms of many of the operators under consideration.

PROPOSITION 2.4. *If S is an operator on $\mathcal{H}$, and the vectors $S\phi_0, S\phi_1, S\phi_2, \ldots,$ are mutually orthogonal, then* $\|S\| = \sup\{\|S\phi_n\| : n = 0, 1, 2, \ldots\}$.

Let D denote the diagonal operator on $\mathcal{H}$ determined by the equations

$$D\phi_n = \exp[-n^2]\phi_n, \qquad n = 0, 1, 2, \ldots, \tag{2.5}$$

and for each positive integer r let

$$\lambda(r) = \exp[r + \mu(r)^2]. \tag{2.6}$$

The following lemma is crucial in the determination of Lat T.

LEMMA 2.7. *As $r \to \infty$,*

$$\lambda(r)T^{\mu(r)+1} \to \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}$$

in the operator norm.

PROOF. For each r,

$$\lambda(r)T^{\mu(r)+1} = \begin{bmatrix} \lambda(r)A^{\mu(r)+1} & \lambda(r)\sum_{j=0}^{\mu(r)} A^j C B^{\mu(r)-j} \\ 0 & \lambda(r)B^{\mu(r)+1} \end{bmatrix}.$$

We deal with the matrix components of $\lambda(r)T^{\mu(r)+1}$ separately.

From (2.1) it follows that for any positive integer j

$$\begin{aligned} A^j\phi_n &= \exp[(n-j)^2 - n^2]\phi_{n-j} \quad \text{if } n \geqslant j, \\ &= 0 \quad \text{otherwise}, \\ B^j\phi_n &= \exp[2(n-j)^2 - 2n^2]\phi_{n-j} \quad \text{if } n \geqslant j, \\ &= 0 \quad \text{otherwise}. \end{aligned} \tag{2.8}$$

Therefore both $A^{\mu(r)+1}$ and $B^{\mu(r)+1}$ satisfy the conditions of (2.4), and so

$$\|A^{\mu(r)+1}\| = \exp[-(\mu(r)+1)^2] \quad \text{and} \quad \|B^{\mu(r)+1}\| = \exp[-2(\mu(r)+1)^2].$$

It follows that

$$\|\lambda(r)A^{\mu(r)+1}\| = \exp\left[r + \mu(r)^2 - (\mu(r)+1)^2\right] \to 0 \quad \text{as } r \to \infty,$$

$$\|\lambda(r)B^{\mu(r)+1}\| = \exp\left[r + \mu(r)^2 - 2(\mu(r)+1)^2\right] \to 0 \quad \text{as } r \to \infty.$$

Consideration of the remaining nonzero term is more complicated. Let E_r denote $\sum_{j=0}^{\mu(r)} A^j C B^{\mu(r)-j}$. We show that $\lambda(r)E_r \to D$ in the operator norm as $r \to \infty$.

For each positive integer s let C_s denote the rank one operator $\exp[-s](\phi_0 \otimes \phi_{\mu(s)})$, and let $E_{r,s}$ denote $\sum_{j=0}^{\mu(r)} A^j C_s B^{\mu(r)-j}$. Clearly $C = \sum_{s=1}^{\infty} C_s$ and $E_r = \sum_{s=1}^{\infty} E_{r,s}$, where the convergence is in the operator norm.

Now $A^j C_s B^{\mu(r)-j}\phi_n = \exp[-s] < B^{\mu(r)-j}\phi_n, \phi_0 > A^j\phi_{\mu(s)}$, and by (2.8)

$$\left\langle B^{\mu(r)-j}\phi_n, \phi_0\right\rangle = \exp\left[-2n^2\right] \quad \text{if } n = \mu(r) - j,$$
$$= 0 \quad \text{otherwise},$$

and

$$A^j\phi_{\mu(s)} = \exp\left[(\mu(s)-j)^2 - \mu(s)^2\right]\phi_{\mu(s)-j} \quad \text{if } j \leqslant \mu(s),$$
$$= 0 \quad \text{otherwise}.$$

Thus it follows that

$$E_{r,s}\phi_n = \exp\left[-s - 2n^2 + (\mu(s) - \mu(r) + n)^2 - \mu(s)^2\right]\phi_{\mu(s)-\mu(r)+n} \quad \text{if } n \in I_{r,s},$$
$$= 0 \quad \text{otherwise}, \tag{2.9}$$

where $I_{r,s}$ denotes the set of nonnegative integers n for which $\mu(r) - \mu(s) \leqslant n \leqslant \mu(r)$. Furthermore $E_{r,s}$ satisfies the conditions of (2.4).

Now $[-s - 2n^2 + (\mu(s) - \mu(r) + n)^2 - \mu(s)^2]$ is a quadratic function of n which achieves its supremum when $n = \mu(s) - \mu(r)$. However $\mu(s) - \mu(r) \in I_{r,s}$ only if $r = s$. In any case the supremum over the interval $I_{r,s}$ is attained at one of its endpoints. Thus

$$\begin{aligned}\|E_{r,s}\| &= \|E_{r,s}\phi_{\mu(r)-\mu(s)}\| \\ &= \exp\left[-s - 2(\mu(r) - \mu(s))^2 - \mu(s)^2\right] \quad \text{if } s \leqslant r, \\ &= \|E_{r,s}\phi_{\mu(r)}\| = \exp\left[-s - 2\mu(r)^2\right] \quad \text{if } s > r.\end{aligned}$$

It follows that if $s < r$ then

$$\begin{aligned}\|\lambda(r)E_{r,s}\| &= \exp\left[r + \mu(r)^2 - s - 2(\mu(r) - \mu(s))^2 - \mu(s)^2\right] \\ &= \exp\left[r - s - \mu(r)^2 + 4\mu(r)\mu(s) - 3\mu(s)^2\right] \\ &< \exp\left[r - \mu(r)(\mu(r) - 4\mu(s))\right] \\ &< \exp\left[r - \mu(r)\right] \quad \text{by (2.2)}.\end{aligned}$$

Therefore $\|\sum_{s=1}^{r-1} \lambda(r)E_{r,s}\| < (r-1)\exp[r - \mu(r)] \to 0$ as $r \to \infty$.

Similarly, if $s > r$ then

$$\begin{aligned}\|\lambda(r)E_{r,s}\| &= \exp\left[r + \mu(r)^2 - s - 2\mu(r)^2\right] \\ &= \exp\left[r - s - \mu(r)^2\right].\end{aligned}$$

Therefore $\|\sum_{s=r+1}^{\infty} \lambda(r)E_{r,s}\| < \exp[-\mu(r)^2] \to 0$ as $r \to \infty$.

Thus to show that $\lambda(r)E_r \to D$, it is sufficient to show that $\lambda(r)E_{r,r} \to D$.

From (2.9) it follows that if $n \leqslant \mu(r)$ then

$$E_{r,r}\phi_n = \exp\left[-r - n^2 - \mu(r)^2\right]\phi_n,$$

and hence $\lambda(r)E_{r,r}\phi_n = \exp[-n^2]\phi_n = D\phi_n$. Since $E_{r,r}\phi_n = 0$ if $n \geqslant \mu(r)$, and since $\exp[-n^2] \to 0$ as $n \to \infty$, it follows that $\lambda(r)E_{r,r} \to D$ in the operator norm as $r \to \infty$. This completes the proof of the lemma.

3. The invariant subspaces. For each nonnegative integer n let $\mathfrak{M}_n$ denote the subspace of $\mathcal{H}$ spanned by the vectors $\phi_0, \phi_1, \phi_2, \ldots, \phi_n$. The subspaces $\mathfrak{M}_n$ satisfy the inclusion relations:

$$\{0\} \subset \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \cdots \subset \mathfrak{M}_n \subset \cdots,$$

and the lattice join of all the subspaces $\mathfrak{M}_n$ is $\mathcal{H}$. Let $\mathfrak{L}$ denote the following set of subspaces of $\mathcal{H}$:

$$\mathfrak{L} = (\{0\}, \mathfrak{M}_n \oplus \{0\}, \mathcal{H} \oplus \{0\}, \mathcal{H} \oplus \mathfrak{M}_n, \mathcal{H}, n = 0, 1, 2, \ldots).$$

Clearly each subspace in $\mathfrak{L}$ is invariant under T. We shall show that these are the only subspaces invariant under T.

THEOREM 3.1. Lat $T = \mathfrak{L}$.

PROOF. We have already remarked that $\mathfrak{L} \subseteq \operatorname{Lat} T$, so it remains to show that $\operatorname{Lat} T \subseteq \mathfrak{L}$.

For each vector $f \oplus g$ in $\mathcal{H}$, let $\mathcal{C}_T(f \oplus g)$ denote the smallest subspace of $\mathcal{H}$ containing $f \oplus g$ and invariant under T. The subspace $\mathcal{C}_T(f \oplus g)$ is called the cyclic subspace generated by T and $f \oplus g$, and is the (closed, linear) span of the vectors $T^n(f \oplus g)$, $n = 0, 1, 2, \ldots$. Since each invariant subspace is the span of cyclic subspaces and since $\mathfrak{L}$ is a complete chain of subspaces, it is sufficient to show that each cyclic subspace is in $\mathfrak{L}$.

For each nonzero vector h in $\mathcal{H}$ define $\deg h$, the degree of h, by $\deg h = \sup\{n: \langle h, \phi_n\rangle \neq 0\}$. If $\langle h, \phi_n\rangle \neq 0$ for infinitely many n, define $\deg h = \infty$.

First consider cyclic subspaces of the type $\mathcal{C}_T(f \oplus 0)$, with $f \neq 0$. Now the restriction of T to $\mathcal{H} \oplus \{0\}$ is A, so $\mathcal{C}_T(f \oplus 0) = \mathcal{C}_A(f) \oplus \{0\}$. By (2.1) A is a backward weighted shift with square-summable, monotonically decreasing weights, so A is unicellular [**4**, problem 151]. In fact, $\mathcal{C}_A(f) = \mathfrak{M}_n$ if $\deg f = n$, and $\mathcal{C}_A(f) = \mathcal{H}$ if $\deg f = \infty$. Thus $\mathcal{C}_T(f \oplus 0) = \mathfrak{M}_n \oplus \{0\}$ or $\mathcal{H} \oplus \{0\}$.

Now consider $\mathcal{C}_T(f \oplus g)$, with $g \neq 0$. We shall use the easily established proposition that $f' \oplus g' \in \mathcal{C}_T(f \oplus g)$ implies $\mathcal{C}_T(f' \oplus g') \subseteq \mathcal{C}_T(f \oplus g)$. Suppose that $\deg g = n$. Write $f' \oplus g' = T^n(f \oplus g)$, and $f'' \oplus g'' = T(f' \oplus g') = T^{n+1}(f \oplus g)$. Since $g' = B^n g$ and $g'' = B^{n+1}g$, $\deg g' = 0$ and $g'' = 0$. Also $\deg f'' = \infty$. To see this suppose $\deg f'' < \infty$. Then $\langle f'', \phi_{\mu(m)}\rangle = 0$ for all sufficiently large m. Now

$$\langle f'', \phi_{\mu(m)}\rangle = \langle Af' + Cg', \phi_{\mu(m)}\rangle,$$

$$\langle Af', \phi_{\mu(m)}\rangle = \exp\left[-1 - 2\mu(m)\right]\langle f', \phi_{\mu(m)+1}\rangle,$$

and

$$\langle Cg', \phi_{\mu(m)}\rangle = \exp[-m]\langle g', \phi_0\rangle.$$

Therefore $\langle f', \phi_{\mu(m)+1}\rangle = -\exp[1 + 2\mu(m) - m]\langle g', \phi_0\rangle$ for all sufficiently large m. Since $\exp[1 + 2\mu(m) - m] \to \infty$ as $m \to \infty$ and $\langle g', \phi_0\rangle \neq 0$, it follows that $\langle f', \phi_{\mu(m)+1}\rangle \to \infty$ as $m \to \infty$. This is impossible, so we conclude that $\deg f'' = \infty$. Thus $\mathcal{K} \oplus \{0\} = \mathcal{C}_T(f'' \oplus 0) \subseteq \mathcal{C}_T(f \oplus g)$, and it follows that $h \oplus g \in \mathcal{C}_T(f \oplus g)$ for all h, and $\mathcal{K} \oplus \mathcal{C}_B(g) \subseteq \mathcal{C}_T(f \oplus g)$. Now B is also a backward weighted shift with square-summable, monotonically decreasing weights, so B is also unicellular [**4**, problem 151]. In fact, since $\deg g = n$, $\mathcal{C}_B(g) = \mathfrak{M}_n$. Since $\mathcal{C}_T(f \oplus g) \subseteq \mathcal{K} \oplus \mathfrak{M}_n$, it follows that $\mathcal{C}_T(f \oplus g) = \mathcal{K} \oplus \mathfrak{M}_n$.

Now suppose $\deg g = \infty$. By (2.7) $Dg \oplus 0 \in \mathcal{C}_T(f \oplus g)$, and since $\deg Dg = \infty$, it follows that $\mathcal{K} \oplus \{0\} = \mathcal{C}_T(Dg \oplus 0) \subseteq \mathcal{C}_T(f \oplus g)$. Again we conclude that $h \oplus g \in \mathcal{C}_T(f \oplus g)$ for each h, and $\mathcal{K} \oplus \mathcal{C}_B(g) \subseteq \mathcal{C}_T(f \oplus g)$. Since $\deg g = \infty$, $\mathcal{C}_B(g) = \mathcal{K}$, and so $\mathcal{C}_T(f \oplus g) = \mathcal{K} \oplus \mathcal{K} = \mathcal{K}$.

It has been shown that for each nonzero vector $f \oplus g$ in $\mathcal{K}$, $\mathcal{C}_T(f \oplus g) \subseteq \mathfrak{L}$. It follows that Lat $T \subseteq \mathfrak{L}$ and the theorem is proved.

COROLLARY 3.2. *$\omega + \omega + 1$ is attainable.*

PROOF. The order-type of the chain $\mathfrak{L}$ is $\omega + \omega + 1$, and by the theorem Lat $T = \mathfrak{L}$.

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