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ISOMORPHISMS OF LOCALLY COMPACT GROUPS AND BANACH ALGEBRAS

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ABSTRACT. If G is a locally compact group, then $\mathrm{UBC}_r(G)^*$, the dual of the space of bounded right uniformly continuous complex-valued functions on G, with the Arens product is a Banach algebra. We prove in this paper a result that will have as a consequence the following: Let G_1, G_2 be locally compact groups. Then the Banach algebras $\mathrm{UBC}_r(G_1)^*$ and $\mathrm{UBC}_r(G_2)^*$ are isometric isomorphic if and only if G_1 and G_2 are topologically isomorphic.

1. Introduction. J. Wendel proved in [7] that if G_1 and G_2 are locally compact groups such that the group algebras $L_1(G_1)$ and $L_1(G_2)$ are isometric isomorphic, then G_1 and G_2 are topologically isomorphic. Later B. Johnson [3] proved that the same conclusion holds when the group algebras are replaced by the measure algebras $M(G_1)$, $M(G_2)$ of G_1 and G_2 respectively.

Let $\mathrm{UBC}_r(G)$ denote the Banach space of bounded right uniformly continuous complex-valued functions on a locally compact group G (see [4, p. 275]) with the supremum norm. On the dual Banach space, $\mathrm{UBC}_r(G)^*$, we may define a product by $\langle m \odot n, f\rangle = \langle m, n_l(f)\rangle$, where $n_l(f)(x) = \langle n, l_x f\rangle$, $l_x f(y) = f(xy)$, for any $m, n \in \mathrm{UBC}_r(G)^*$, $f \in \mathrm{UBC}_r(G)$, and $x, y \in G$. Then $\mathrm{UBC}_r(G)^*$ with respect to this product is a Banach algebra (see [4, p. 275]). In this paper we shall prove a theorem (Theorem 1) which will have as consequences both Johnson's result [3, Corollary 1] and the following:

THEOREM 2. *Let G_1, G_2 be locally compact groups. Then the Banach algebras $\mathrm{UBC}_r(G_1)^*$ and $\mathrm{UBC}_r(G_2)^*$ are isometric isomorphic if and only if G_1 and G_2 are topologically isomorphic.*

2. Some technical lemmas. Throughout this section, G will denote a locally compact group with a fixed left Haar measure λ. Integration with respect to λ will be denoted by $\int_G \ldots dx$. If f, h are complex-valued measurable functions defined λ a.e. on G, let

$$\tilde{f}(x) = f(x^{-1}), \qquad l_x f(y) = f(xy), \qquad r_x f(y) = f(yx),$$

$$(f * h)(x) = \int_G f(y)h(y^{-1}x)\,dy, \qquad (\mu * h)(x) = \int_G h(y^{-1}x)\,d\mu(y)$$

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for any $x, y \in G$ and regular Borel measure μ on G whenever the formula makes sense.

Let $C(G)$ denote the space of bounded continuous complex-valued functions on G with the supremum norm $\|\cdot\|_u$, and let $C_0(G)$ be the closed subspace of $C(G)$ consisting of all functions vanishing at infinity. A closed linear subspace X of $C(G)$ is *left introverted* (see Day [1, p. 540]) if $l_a(X) \subseteq X$ for each $a \in G$, and for each $m \in X^*$, $f \in X$, the function $m_l(f)$ on G defined by $m_l(f)(x) = m(l_x f)$, $x \in G$, is also in X. In this case, the Arens multiplication on X^* defined by $\langle n \odot m, f\rangle = \langle n, m_l(f)\rangle$ for each $f \in X$, $n, m \in X^*$ makes sense. Furthermore, X^* with this multiplication is a Banach algebra (see [1, §6]). Examples of left introverted subspaces of $C(G)$ include $C_0(G)$, $\mathrm{UBC}_r(G)$, and the space of almost periodic (resp. weakly almost periodic) continuous functions on G, denoted by $\mathrm{AP}(G)$ (resp. $\mathrm{WAP}(G)$). In the case of $C_0(G)^* = M(G)$, the multiplication $\odot$ on $M(G)$ is precisely the convolution of measures as defined in [4, p. 266].

If $a \in G$, δ_a will denote either the point-measure at a in $M(G)$, or the point-evaluation linear functional in X^* when X is a subspace of $C(G)$.

LEMMA 1. *Let $m \in C_0(G)^*$. Then m has a unique norm preserving extension to a continuous linear functional on $C(G)$.*

PROOF. We may assume that $\|m\| = 1$. Let μ be a regular Borel measure on G such that $m(f) = \int_G f(x)\, d\mu(x)$ for each $f \in C_0(G)$. Define $\tilde{m}(f) = \int_G f(x)\, d\mu(x)$ for all $f \in C(G)$. Then $\tilde{m} \in C(G)^*$, $\tilde{m}$ extends m, and $\|\tilde{m}\| = 1$.

If $n \in C(G)^*$ is another extension of m, and $\|n\| = 1$, let F denote the set of linear combinations of point evaluations on G in the unit ball of $C(G)^*$. If n is not in the weak*-closure of F, then by [2, p. 417, Theorem 10] there exist $f \in C(G)$, a constant c and $\varepsilon > 0$ such that

$$\mathrm{Re}\langle f, \phi\rangle \leqslant c - \varepsilon \leqslant c \leqslant \mathrm{Re}\langle f, n\rangle$$

for all $\phi \in F$. Now if $a \in G$, and $f(a) = re^{i\theta}$, let $\phi = e^{-i\theta}\delta_a$; then $\langle \phi, f\rangle = r \leqslant c - \varepsilon$. Hence $|f(a)| = r \leqslant c - \varepsilon$. In particular, $\|f\|_u \leqslant c - \varepsilon$.

However $c \leqslant \mathrm{Re}\langle f, n\rangle \leqslant |\langle f, n\rangle| \leqslant \|n\|\,\|f\|_u \leqslant c - \varepsilon$ which is impossible. Hence n is in the weak*-closure of F.

Let m_α be a net in F, such that $m_\alpha(f) \to n(f)$ for each $f \in C(G)$. Let $q(m_\alpha)$ denote the restriction of m_α to $C_0(G)$. Then $\|q(m_\alpha)\| \leqslant 1$, and $q(m_\alpha)(f) \to \int_G f\, d\mu$ for each $f \in C_0(G)$. Hence $\|q(m_\alpha)\| \to \|\mu\| = 1$. It follows from [6, Theorem 3.9] that $\langle m_\alpha, f\rangle$ also converges to $\int_G f\, d\mu = \tilde{m}(f)$ for each $f \in C(G)$. In particular, $n = \tilde{m}$.

LEMMA 2. *Let X be a left introverted closed subspace of $C(G)$ containing $C_0(G)$. If $m \in X^*$ such that m_l: $X \to X$ is an isometry, then there exist $x \in G$ and $\lambda \in C$, $|\lambda| = 1$ such that $m_l(f) = \lambda r_x(f)$ for all $f \in X$.*

PROOF. We first assume that $X = C_0(G)$. Let $\mu \in M(G)$ such that $\langle m, f\rangle = \int_G f\, d\mu$ for all $f \in C_0(G)$.

Define: τ: $L_1(G) \to L_1(G)$ by $\tau(f) = \mu * f$, $f \in L_1(G)$. Then τ is an isometry. Clearly $\|\tau(f)\|_1 \leqslant \|f\|_1$ for each $f \in L_1(G)$ since $\|\mu\| = 1$. To prove the reverse inequality, let $f \in L_1(G)$ be fixed. For any $\varepsilon > 0$, choose $h \in C_0(G)$ such that $\|h\|_u = 1$ and $|\int f(x)h(x)\,dx| > \|f\|_1 - \varepsilon$. Then $\|(f * \tilde{h})^{\sim}\|_u > \|f\|_1 - \varepsilon$. Hence

$$\begin{aligned}\|\mu * f\|_1 &= \|\mu * f\|_1 \|\tilde{h}\|_u \geq \|(\mu * f) * \tilde{h}\|_u \\ &= \left\|\mu * \left[(f * \tilde{h})^{\sim}\right]^{\sim}\right\|_u = \left\|m_l((f * \tilde{h})^{\sim})\right\|_u \\ &= \left\|(f * \tilde{h})^{\sim}\right\|_u > \|f\|_1 - \varepsilon.\end{aligned}$$

By [7, Theorem 3], there exist $y \in G$ and $\lambda \in C$, $|\lambda| = 1$, such that $\mu * f = \lambda l_y f$ for all $f \in L_1(G)$. In particular $m_l(f) = (\mu * \tilde{f})^{\sim} = \lambda r_x f$, where $x = y^{-1}$, for all $f \in C(G)$ with compact support. Hence $m_l(f) = \lambda r_x(f)$ for all $f \in C_0(G)$.

The general case follows from Lemma 1 by considering the restriction of m to $C_0(G)$.

If X is a linear subspace of $C(X)$, and $h \in L_1(G)$, we define $\tau(h) \in X^*$ by

$$\tau(h)(f) = \int_G f(x)h(x)\,dx \quad \text{for all } f \in X.$$

LEMMA 3. *If X is a closed left introverted subspace of $C(X)$ consisting of left uniformly continuous functions, then the set*

$$\{m_l(f);\ m = \tau(h),\ h \in L_1(G) \text{ and } f \in X\}$$

is norm dense in X.

PROOF. Let $\{h_\alpha\}$ be a bounded approximate identity in $L_1(G)$ and $m_\alpha = m(h_\alpha)$. Then

$$\|(m_\alpha)_l(f) - f\|_u = \left\|(h_\alpha * \tilde{f})^{\sim} - f\right\|_u = \|h_\alpha * f - f\|_u$$

which converges to zero (see [5, 32.48]).

REMARK. Lemma 3 is false when the functions in X are not left uniformly continuous. For example, if $X = \mathrm{UBC}_r(G)$, then for each $f \in X$, $h \in L_1(G)$, and $m = \tau(h)$, the function $m_l(f) = f * \tilde{h}$ is both left and right uniformly continuous (see [4, 20.16]).

3. The main theorem. We are now ready to prove our main result.

THEOREM 1. *Let G_1, G_2 be locally compact groups. Let X_i, $i = 1, 2$, be a closed left introverted subspace of $C(G_i)$ containing $C_0(G_i)$. If the Banach algebras X_1^* and X_2^* are isometric isomorphic, then G_1 and G_2 are topologically isomorphic.*

PROOF. Let β: $X_1^* \to X_2^*$ be an isometric isomorphism. For $a \in G$ and $m = \delta_a$, we have $m^{-1} = \delta_b$ in X_1^*, where $b = a^{-1}$, and so

$$\|\beta(m)\| = \|m\| = 1 = \|m^{-1}\| = \|\beta(m)^{-1}\|$$

which implies that $\beta(m)_l$ is a linear isometry. An application of Lemma 2 shows that there exist $\gamma(a)$ in G_2 and $\lambda(a) \in C$ such that $\beta(m)_l = \lambda(a) r_{\gamma(a)}$. Then as readily checked, γ is an algebraic isomorphism from G_1 onto G_2. It remains to show that γ is a homeomorphism.

Let $\{x_\alpha\}$ be a net in G_1 converging to the identity e_1. Let $\{\gamma(x_\pi)\}$ be any subnet of the net $\{\gamma(x_\alpha)\}$. By Alaoglu's theorem, we may assume (passing to a subnet if necessary) that the net $\theta_\pi = \beta(\delta_{x_\pi})$ converges to some $\eta \in X_2^*$ in the weak*-topology of X_2^*. Since the net $\{\lambda(x_\pi)\}$ is bounded, we may further assume that the net $\lambda(x_\pi)$ also converges. Let $h \in L_1(G)$, and $m = \tau(h)$ (see Lemma 3), then

$$(\delta_{x_\pi} \odot m)(f) = \int_{G_1} h(x_\pi^{-1}y)f(y)\,dy$$

for each $f \in X_1$. Hence $\|\delta_{x_\pi} \odot m - m\| \leqslant \|l_{z_\pi}h - h\|_1$, where $z_\pi = x_\pi^{-1}$, which converges to zero. This implies that the net $\|\theta_\pi \odot \beta(m) - \beta(m)\| \to 0$ also. In particular,

$$[\,\beta(m)\,](k) = \lim_\pi \theta_\pi \odot \beta(m)(k) = \eta \odot \beta(m)(k)$$

for each $k \in X_2$. So $\beta(m) = \eta \odot \beta(m)$ and hence $m = \beta^{-1}(\eta) \odot m$. Consequently $m_l(f)(e_1) = \beta^{-1}(\eta)(m_l(f))$ for each $f \in X_1$ and $m = \tau(h)$, $h \in L_1(G)$. Since $C_0(G_1) \subseteq X_1$ and is left introverted, Lemma 3 implies that $\beta^{-1}(\eta)$ agrees with $\delta(e_1)$ on $C_0(G_1)$. Since $1 = \|\delta_{e_1}\| \leqslant \|\beta^{-1}(\eta)\| = \|\eta\| \leqslant 1$, it follows from Lemma 2 that $\beta^{-1}(\eta)$ and δ_{e_1} agree on X_1 also. In particular $\eta = \delta_{e_2}$. Hence the net $\{\theta_\pi\}$ converges to δ_{e_2} in the weak*-topology of X_2^*. This implies that $\lambda(x_\pi) \to 1$ and the net $\delta_{\gamma(x_\pi)}$ converges to δ_{e_2} in the weak*-topology of X_2^*. Since $C_0(G_2) \subseteq X_2$, $\gamma(x_\pi)$ converges to e_2. Since $\{\gamma(x_\pi)\}$ is an arbitrary subnet of $\{\gamma(x_2)\}$, the net $\{\gamma(x_\alpha)\}$ must also converge to e_2. Therefore γ is continuous. Similarly γ^{-1} is also continuous. The theorem is established.

REMARKS. 1. Theorem 1 yields Johnson's result [3, Corollary] when $X_i = C_0(G_i)$, $i = 1, 2$.

2. Theorem 1 is false without the hypothesis: $C_0(G_1) \subseteq X_1$, for example, if G_1 is any locally compact noncompact group, and G_2 is the set of multiplicative linear functionals on $\mathrm{AP}(G_1)$ with the Arens multiplication. Then G_2 is a compact group. Let $X_1 = \mathrm{AP}(G_1)$ and $X_2 = C(G_2)$. Then $X_1 \cap C_0(G_1) = \{0\}$. Define $J\colon X_1 \to X_2$ by $(Jf)(\phi) = \phi(f)$ for each $f \in X_1$, $\phi \in G_2$. Then J^* is an isometry and an algebra isomorphism from X_2^* onto X_1^*.

REFERENCES

1. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.

2. N. Dunford and J. T. Schwartz, *Linear operators*. I, Interscience, New York, 1958.

3. B. E. Johnson, *Isometric isomorphisms of measure algebras*, Proc. Amer. Math. Soc. **15** (1964), 186–188.

4. E. Hewitt and K. Ross, *Abstract harmonic analysis*, vol. 1, Springer-Verlag, Berlin, 1964.

5. ______, *Abstract harmonic analysis*, vol. 2, Springer-Verlag, Berlin, 1970.

6. K. McKennon, *Multipliers, positive functionals, positive-definite functions, and Fourier-Stieltjes transforms*, Mem. Amer. Math. Soc., no. 111, 1971.

7. J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. **2** (1952), 251–261.

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