ON AUTONOMOUS CONTROL SYSTEMS ON CERTAIN MANIFOLDS

CHAO-CHU LIANG

Abstract. Let $M^n$ be a compact $C^\infty$ manifold, $n > 4$, admitting a vector field with every orbit a circle. Then there exists a completely controllable set $S$ consisting of two nonsingular $C^\infty$ vectors $X$ and $Y$ such that every orbit of $X$ is a circle.

An autonomous control system on a smooth manifold $M$ is the same as a set of vector fields on $M$. A set $S$ of vector fields on a smooth manifold $M$ is said to be controllable if for every pair $(m, m')$ of points of $M$ there exists a trajectory of $S$ from $m$ to $m'$. Here a trajectory of $S$ is a curve which is an integral curve (orbit) of some $X \in S$ or a finite concatenation of such curves such that a trajectory of $S$ run in reverse is not allowed. (We refer the readers to [2] for details.)

In [2], N. Levitt and H. J. Sussmann showed that on every connected paracompact manifold of class $C^k$, $2 < k < \infty$, or $k = \omega$, there exists a completely controllable set $S$ consisting of two vector fields of class $C^{k-1}$.

For simplicity, we assume that all the manifolds, vector fields, etc., considered here are of class $C^\infty$.

A manifold $M$ is called closed if it is compact and without boundary. Let $D^k$ denote the $k$-dimensional disk and $S^{k-1}$ its boundary.

The purpose of this paper is to prove the following theorem:

**Theorem.** If a connected closed $n$-dimensional manifold $M^n$, $n > 4$, admits a vector field $X_0$ with every orbit a (nondegenerate) circle, then there exists a completely controllable set $S$ consisting of two nonsingular vectors $X$ and $Y$ such that every orbit of $X$ is a circle and $Y$ has finitely many closed orbits.

We first give a brief sketch of the proof.

According to [1], $M$ can be decomposed as a union of round handles $R_k = S^1 \times D^k \times D^{n-k-1}$. Each round $k$-handle $R_k$ is supplied with a vector field

$$V = \frac{d}{dt} - \sum x_i \frac{\partial}{\partial x_i} + \sum y_j \frac{\partial}{\partial y_j},$$

where $(t, x, y) \in S^1 \times D^k \times D^{n-k-1}$. A point $p \in R_k$ can be moved along a trajectory of $V$ arbitrarily close to the closed orbit $S^1 \times 0 \times 0$ if and only if $p \in S^1 \times D^k \times 0$. Modifying the vector field $V$ on each $R_k$, we will get a...
nonsingular vector field $W$ such that for $k > 0$ (respectively $k = 0$) any trajectory of $W$ approaching $S^1 \times 0 \times 0$ (respectively any trajectory of $W$) meets $S^1 \times (x_1$-axis) (respectively $S^1 \times (y_1$-axis)) in $R_k$. By patching up the vector fields $W$ on the $R_k$'s as in [1], we construct a nonsingular vector field $Y$ on $M$ with finitely many closed orbits $\{C_i\}$, where $C_i$ corresponds to $S^1 \times 0 \times 0$ on each $R_k$. By using the standard transversality argument, we may assume that near $C_i$ for each $t \in S^1$ the $x_1$-axis (or $y_1$-axis) forms part of an orbit of $X$. Then we construct a sequence of diffeomorphisms $f_1, \ldots, f_{n-1}$ of $M$ to itself such that the $C_i$'s are connected by the orbits of the vector field $X = f_{(N-1)} \ldots f_1(X_0)$. The set $\{X, Y\}$ is showed to be completely controllable.

Let $V$ denote the vector
\[
d/dt - \sum_{i=1}^k x_i \partial/\partial x_i + \sum_{j=1}^{n-k-1} y_j \partial/\partial y_j
\]
on $R_k = S^1 \times D^k \times D^{n-k-1}$, where the $x_i$'s and $y_j$'s denote the standard coordinate functions on $R^k$ and $R^{n-k-1}$ respectively.

**Lemma 1.** For $k > 0$, there exists a vector field $W$ on $R_k$ such that $W$ has $S^1 = S^1 \times 0 \times 0$ as its only closed orbit, and $S^1 \times (x_1$-axis $- 0)$ is reachable from every trajectory except $S^1$.

**Proof.** Assume that $k > 1$, let $B_j \subseteq D^k$ be the 2-dimensional disk spanned by $x_1$-axis and $x_j$-axis, $1 < j < k$. We write $rB_j$ for the concentric disk of radius $r$. For small $\theta_0 > 0$, we may construct a diffeomorphism $g_{\theta_j} : B_j \rightarrow B_j$ for each $0 < \theta < \theta_0$ such that $g_{\theta_j}$ fixes the complement of $\frac{1}{4}B_j$, $g_{\theta_j}[\frac{1}{4}B_j] = -r$ by a degree of $\theta$, and $G_j(\theta, x_1, x_j) = g_{\theta_j}(x_1, x_j), 0 < \theta < \theta_0$, is an isotopy with $g_0 =$ identity. Then we define $g_{\theta_j}$ for arbitrary $\theta > 0$ by $g_{\theta_j} = g_{p\theta_0}g_{\theta_0}^p$, where $\theta = p\theta_0 + r$ with $p$ an integer and $r > 0$. The diffeomorphism $g_{\theta_j}$ induces a diffeomorphism $h_{\theta_j}$ on $D^k \times D^{n-k-1}$ by fixing the remaining coordinates. We also write
\[
h_{\theta_j}(\theta, x_1, \ldots, x_k, y_1, \ldots, y_{n-k-1}) = h_{\theta_j}(x_1, \ldots, y_{n-k-1}).
\]

We construct an isotopy
\[
F_j : [4j\pi, (4j + 1)\pi] \times D^k \times D^{n-k-1} \rightarrow D^k \times D^{n-k-1}
\]
as follows:
\[
F_j(t, x, y) = \begin{cases} 
H_j(t - 4j\pi, x, y) & \text{for } t \in [4j\pi, (4j + 1)\pi], \\
H_j(\pi, x, y) & \text{for } t \in [(4j + 1)\pi, (4j + 2)\pi], \\
H_j((4j + 3)\pi - t, x, y) & \text{for } t \in [(4j + 2)\pi, (4j + 3)\pi], \\
H_j(0, x, y) & \text{for } t \in [(4j + 3)\pi, (4j + 4)\pi].
\end{cases}
\]

The map $F_j$ induces a diffeomorphism $K_j$ from $[4j\pi, (4j + 1)\pi] \times D^k \times D^{n-k-1}$ to itself, where $K_j(t, x, y) = (t, F_j(t, x, y))$. By gluing together $K_j$, $2 < j < k$, on the
common boundaries, we obtain a diffeomorphism $K$ of $[8\pi, 4\pi] \times D^k \times D^{n-k-1}$ to itself. Identifying $8\pi$ with $4\pi$, we thus have a diffeomorphism $\tilde{K}$ of $S^1 \times D^k \times D^{n-k-1}$ (geometrically, $\tilde{K}$ is given by twisting $\frac{1}{2}B_j$ along $S^1$ by $180^\circ$, and then twisting it back $180^\circ$ successively for $2 < j < n - 1$). We define $W$ to be $\tilde{K}_* (V)$ on $R_k$. Q.E.D.

The same proof yields the following lemma:

**Lemma 2.** There exists a nonsingular vector field $W$ on $R_0 = S^1 \times D^0 \times D^{n-1} = S^1 \times D^{n-1}$ with every trajectory except $S^1$ leaving $R_0$, and every trajectory meets $S^1 \times (y_1$-axis).

For a round $k$-handle $R_k = S^1 \times D^k \times D^{n-k-1}$, we write $\partial_- R_k = S^1 \times D^k \times D^{n-k-2}$ [1, p. 42].

**Proof of the Theorem.** Since $M$ supports a nonsingular vector field $X_0$, its Euler number vanishes. According to [1], for $n > 4$, $M$ admits a round handle decomposition, that is, $M$ can be written as $R_0 + \cdots + R_0^0 + \cdots + R_{n-1}^0 + \cdots + R_{n-1}^{n-1}$, where each $R_i^t$ denote a round $k$-handle attached to the boundary of the stuff on the left to it, successively (using $\partial_- R_k$ as the attaching region at each stage), [1, p. 43]. Near $\partial_- R_k$, when $k > 0$, $W$ points inwards (into $R_k^t$). Hence we may use the argument in [1, pp. 52-53] to patch up the $W$'s to construct a vector field $Y$ on $M$ with finitely many closed orbits ({$C_j$}), corresponding to the core $S^1 = S^1 \times 0 \times 0$ in $R_k^t$. Furthermore, by the standard transversality argument, we may assume that the orbits of $X_0$ meet each $C_j$ transversely. Therefore, near each $C_j^t = S^1$, for each $t \in S^1$ the $x_1$-axis (or $y_1$-axis when $k = 0$) forms part of an orbit of $X_0$.

Recall that $M = R_0 + \cdots + R_0^0 + \cdots + R_{n-1}^{n-1}$ with $C_j \subseteq R_j$. We order the $C_j$'s from left to right in this decomposition, and denote them by ({$C_j$})$_{j=0}^N$.

Now we are going to construct a sequence of diffeomorphisms $f_j$, $0 < j < N - 1$, from $M$ to itself such that $C_j$ and $C_{j+1}$ are connected by an orbit $\gamma_j$ of $f_j(X_j) = X_{j+1}$, and $f_j(\gamma_i)$ = $\gamma_i$ when $i < j$. Let $\beta_j$ be an orbit of $X_j$ meeting $C_j$, and $p$ and $a$ point on $\beta_j$ but not on $C_j$ (such a point exists, because of the transversality). We embed a curve $\partial$: $[0, 1] \rightarrow M$ with $\partial(0) = p$ and $\partial(1) = p'$, a point on $C_j$. Since $n > 4$, we may assume that $\partial([0, 1])$ does not intersect any of the other $C_i$'s and $\gamma_i$'s with $i < j$. Let $U_j$ be a tubular neighborhood of $\partial([0, 1])$ in $M$ with $U_j \cap C_{j+1} = l$, a line segment, and $U_j$ is disjoint from all the other $C_i$'s and $\gamma_i$'s with $i < j$. As in [1, pp. 44-45], we construct an isotopy $F_j$ with support in $U_j$ from the identity to a diffeomorphism $f_j$ with $f_j(p) = p'$ (geometrically, we drag $p$ to $p'$ along the path $\partial$). The orbit $\gamma_j = f_j(\beta_j)$ of the vector field $f_j(X_j) = X_{j+1}$ connects $C_j$ and $C_{j+1}$. We apply this argument successively to get a vector field $X_N$ with an orbit $\gamma_j$ connecting $C_j$ and $C_{j+1}$ for each $j$ with $0 < j < N - 1$.

By using the transversality argument again if necessary, we perturb $X_N$ near each $C_j$ to get a vector field $X$ (also with every orbit a circle) such that near $C_j^t$ for each $t \in S^1$ the $x_1$-axis (or $y_1$-axis when $k = 0$) forms parts of an orbit of $S$. We claim that ({$X, Y$}) forms a completely controllable system. Given any two points $m, m'$ on $M$. From the description of $Y$, we see that $m'$ must be on a
trajectory coming out from some $R_i$. On the other hand, $m$ is on a trajectory approaching some $C'_k \subseteq R'_k$ with $k > 0$. We can reach $C'_k$ from $C'_k$ by a sequence of the trajectories $\gamma_j$ of $X$ constructed above and some closed orbits of $Y$. From Lemma 1, Lemma 2, and the last paragraph, we see that $C'_k$ is reachable from $m$, and $m'$ is reachable from $C'_0$. Hence $m'$ is reachable from $m$ by trajectories (in the positive direction) of the system $(X, Y)$. Q.E.D.

Any closed connected manifold $M^n$, $n > 4$, which is the total space of an $S^1$-bundle satisfies the conclusion of the theorem. For example, the odd-dimensional sphere $S^{2k+1}$ is the total space of the Hopf bundle $S^1 \to S^{2k+1} \to \mathbb{C}P(k)$, where $\mathbb{C}P(k)$ denotes the $k$-dimensional complex projective space.

For $n = 3$, the proof shows that if $M^3$ satisfies the additional condition that it admits a round handle decomposition, then the conclusion also holds. For example, $S^3$ can be written as $S^1 \times D^2 + D^2 \times S^1$, one round 0-handle and one round 2-handle.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045