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TRIANGLES IN ARRANGEMENTS OF LINES. II

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ABSTRACT. We show that given n lines in the real projective plane, no $n-1$ of which are concurrent, the number p_3 of triangular regions formed is at most $\frac{2}{5}n(n-1)$, equality being possible.

We also show that if $n \geqslant 6$ then $p_3 \leqslant \frac{7}{18}n(n-1) + \frac{1}{3}$. Grünbaum has conjectured $p_3 \leqslant \frac{1}{3}n(n-1)$.

1. Introduction. Let A be an arrangement of n lines in the real projective plane $\mathbf{P}_2$, and let p_k denote the number of k-sided regions determined by the lines. It is assumed that A is not a pencil. In [1] Grünbaum raises the question of finding an upper bound for p_3 and conjectures that

$$p_3 \leqslant \left[\tfrac{1}{3}n\left(2\left[n/2\right] - 1\right)\right] \leqslant \tfrac{1}{3}n(n-1)$$

for $n \geqslant 16$, and gives R. J. Canham's proof of this when A is simple (no three lines concurrent). The first nontrivial results on the general problem were found by T. O. Strommer [3] who proved among other things that

$$p_3 \leqslant \frac{1}{3}n(n-1) + 2 + \frac{1}{3}\sum_{k \geqslant 4}(k-6)p_k.$$

In [2] we extended these results and obtained some inequalities that implied $p_3 \leqslant \frac{5}{12}n(n-1)$. We now extend these results further, proving two results.

THEOREM 1. *$p_3 \leqslant \frac{2}{5}n(n-1)$ if A is not a near pencil ($n-1$ lines concurrent), and the result is best possible since equality occurs when $n = 6$.*

THEOREM 2. *$p_3 \leqslant \frac{7}{18}n(n-1) + \frac{1}{3}$ if A is not a near pencil.*

REMARK. If A is a near pencil, then $p_3 = 2n - 2$.

2. The proofs.

PROOF OF THEOREM 1. Let A be an arrangement of n lines in $\mathbf{P}_2$, not a pencil or near pencil. Let t_k be the number of points incident with precisely k lines. For a fixed line l of the arrangement, let $t_k(l)$ denote the number of points on l incident with precisely k lines, including l. Let us call a point incident with precisely two lines a simple point, and the other points multiple points. Let $P_1, \ldots, P_m$ be the multiple points, in order, on the line l, with the understanding that $m = 0$ if there are no multiple points on l. We let $I_i = (P_i, P_{i+1})$ called the ith block be the set of simple points on l between P_i and P_{i+1} on l, with the understanding that

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$(P_m, P_{m+1}) = (P_m, P_1)$. In the event that $m = 1$ we have the single block (P_1, P_1). We must therefore allow for the possibility that the endpoints of $I_i = (P_i, P_{i+1})$ are the same point. If $m = 0$ there are no blocks. Each block I_i determines $|I_i| + 1$ line segments $\overline{P_iQ_i}, \overline{Q_1Q_2}, \ldots, \overline{Q_{k-1}Q_k}, \overline{Q_kP_{i+1}}$, where $I_i = \{Q_1, \ldots, Q_k\}$, where $|S|$ denotes the cardinality of the set S. If $I_i = \varnothing$, then there is only the line segment $\overline{P_iP_{i+1}}$. Let $1 \leqslant j \leqslant k$. The line segment $\overline{Q_jQ_{j+1}}$ is the side of at most one triangle, for suppose $\overline{Q_jQ_{j+1}}$ is incident with two triangles. Then the removal of l would result in a digon (a two-sided polygon), and digons only occur in pencils, contradicting the assumption that A is not a near pencil. Thus, if $p_3(l)$ denotes the number of triangles having a side lying on l, then the contribution of I_i to $p_3(l)$ is at most $k + 3 = |I_i| + 3$. If in fact the contribution is less and if $I_i \neq \varnothing$, then we say that I_i is a deficient block. Let D^l denote the number of deficient blocks on l and E^l the number of empty blocks on l. If $I_i = \varnothing$, then the contribution to $p_3(l)$ is only $|I_i| + 2 = 2$. Hence, if $m \geqslant 1$,

$$\begin{aligned} p_3(l) &\leqslant \sum_i \{|I_i| + 3\} - D^l - E^l \\ &= t_2(l) + 3 \sum_{k \geqslant 3} t_k(l) - D^l - E^l. \end{aligned}$$

If $m = 0$,

$$p_3(l) \leqslant t_2(l) = t_2(l) + 3 \sum_{k \geqslant 3} t_k(l) - D^l - E^l,$$

and so, in any case,

$$p_3(l) \leqslant t_2(l) + 3 \sum_{k \geqslant 3} t_k(l) - D^l - E^l.$$

If we let E and D denote respectively the number of empty and deficient blocks on all lines, then taking into account that triangles get counted three times and points on k lines get counted k times, we obtain on summation over l,

$$3p_3 \leqslant 2t_2 + 3 \sum_{k \geqslant 3} kt_k - E - D. \tag{1}$$

We need to consider so-called F-blocks. Let $|I_i| = 2$, $I_i = \{Q_1, Q_2\}$, and suppose that I_i is not deficient. Then we have the situation depicted in Figure 1, with $|I_i| + 3 = 5$ triangles numbered from one to five. We call such a block a type F-block and it forces the existence of a point P incident with at least four lines. Let us give the segments $\overline{Q_1P}$ and $\overline{Q_2P}$ weight one to indicate that they cannot occur in any other block, and give $\overline{P_iP}$ and $\overline{P_{i+1}P}$ weight $\frac{1}{2}$ to indicate that they can each occur in at most two blocks (or possibly $P_i = P_{i+1}$). The sum of these weights is $3 \sum_l F^l = 3F$, where F^l is the number of F-blocks on l, and F is the total number of F-blocks. The sum of these weights cannot exceed $\sum_{k \geqslant 4} 2kt_k$, and so we have

$$\frac{4}{3} \sum_{k \geqslant 4} kt_k \geqslant 2F. \tag{2}$$

Let us now look at a typical block I_i which is not an empty block, a deficient block or an F-block. $I_i = \{Q_1, \ldots, Q_k\}$. See Figure 2. The triangles are marked with the symbol Δ. The points P and P' are each incident with three or more lines,

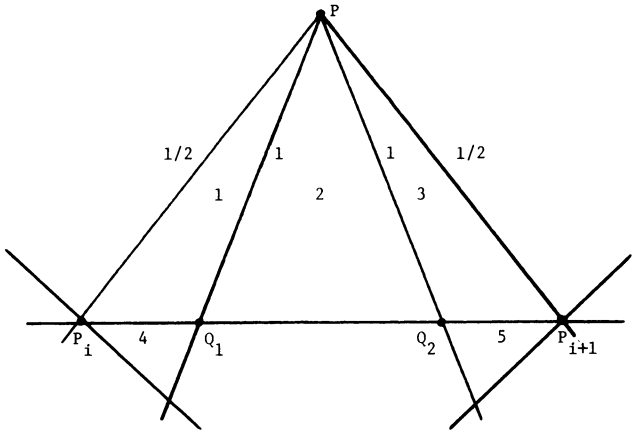

FIGURE 1

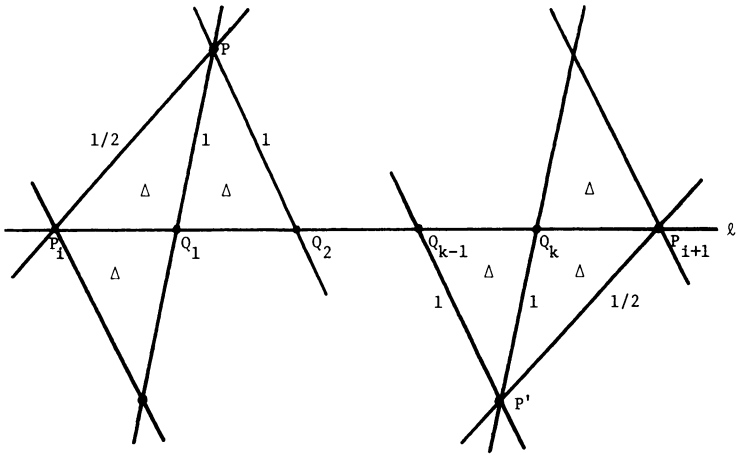

FIGURE 2

and we have given $\overline{P_iP}$ and $P_{i+1}P'$ weight $\frac{1}{2}$ and we have given $\overline{Q_1P}$, $\overline{Q_2P}$, $\overline{Q_{k-1}P'}$ and $\overline{Q_kP'}$ weight one, and block I_i has total weight 5 in the figure drawn. In fact, P and P' could lie on the same side of l and might even coincide. We claim that in all cases the total weight will be at least 4. In Figure 3 we see how 4 can occur. If P_i and P_{i+1} coincide it is to be understood that each occurrence of $\overline{P_iP}$ in Figure 2 or 3 gets weight $\frac{1}{2}$, so that the single edge gets, in fact, a weight of one.

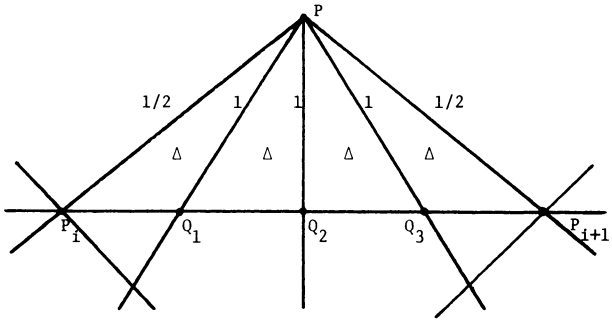

FIGURE 3

The total weight from all of these blocks cannot exceed $\Sigma_{k\geq 3} 2kt_k$, and the number of these blocks on the line l is

$$\sum_{k\geq 3} t_k(l) - F^l - E^l - D^l.$$

Hence

$$\sum_{k\geq 3} 2kt_k \geq 4\sum_l \left\{ \sum_{k\geq 3} t_k(l) - F^l - E^l - D^l \right\}$$
$$= 4\left\{ \sum_{k\geq 3} kt_k - F - E - D \right\},$$

and we have

$$2E + 2D \geq \sum_{k\geq 3} kt_k - 2F,$$

and adding inequality (2), we obtain

$$2E + 2D \geq 3t_3 - \frac{1}{3}\sum_{k\geq 4} kt_k.$$

Combining with (1), we get

$$3p_3 \leq 2t_2 + \frac{15}{2}t_3 + \frac{19}{6}\sum_{k\geq 4} kt_k. \tag{3}$$

Let p_k be the number of k-gons determined by the region. Counting all of the edges, we see that $\Sigma\, kp_k = 2\,\Sigma\, kt_k$, which we rewrite as

$$3p_3 \leq 4t_2 + 6t_3 + 2\sum_{k\geq 4} kt_k - \sum_{k\geq 4} kp_k. \tag{4}$$

Adding $\frac{4}{5}$ of (3) to $\frac{1}{5}$ of (4) we obtain

$$3p_3 \leq \frac{12}{5}t_2 + \frac{36}{5}t_3 + \frac{44}{15}\sum_{k\geq 4} kt_k$$
$$\leq \frac{12}{5}\left\{ t_2 + 3t_3 + \sum_{k\geq 4}\binom{k}{2}t_k \right\} = \frac{6}{5}n(n-1),$$

and $p_3 \leq \frac{2}{5}n(n-1)$. To see that equality can occur for $n = 6$, see Figure 4, and Theorem 1 is proved.

Proof of Theorem 2. Thomas O. Strommer [3] proved

$$p_3 \leq \frac{1}{3}n(n-1) + 4 - \frac{2}{3}t_2 + \sum_{k\geq 4}(k-4)p_k. \tag{5}$$

Adding this to (4) and multiplying the result by $\frac{3}{4}$ we obtain

$$3p_3 \leq \frac{1}{4}n(n-1) + 3 + \frac{5}{2}t_2 + \frac{9}{2}t_3 + \frac{3}{2}\sum_{k\geq 4} kt_k - 3\sum_{k\geq 4} p_k. \tag{6}$$

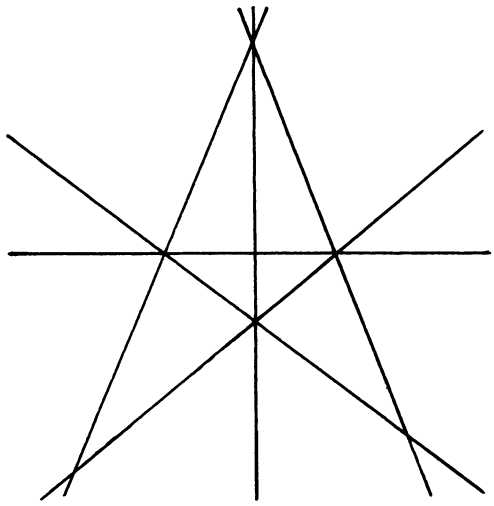

FIGURE 4. $n = 6, p_3 = 12 = \frac{2}{5}n(n-1)$

If we now add $\frac{1}{3}$ of (6) to $\frac{2}{3}$ of (3) we obtain

$$\begin{aligned} 3p_3 &\leqslant \frac{1}{12}n(n-1) + 1 + \frac{13}{6}t_2 + \frac{39}{6}t_3 + \frac{47}{18}\sum_{k\geqslant 4} kt_k \\ &\leqslant \frac{1}{12}n(n-1) + 1 + \frac{13}{6}\left\{t_2 + 3t_3 + \sum_{k\geqslant 4}\binom{k}{2}t_k\right\} \\ &= \frac{14}{12}n(n-1) + 1, \end{aligned}$$

and Theorem 2 follows.

REMARK. Inequality (6) gives a rather curious upper bound on $\Sigma_{k\geqslant 3} p_k$, the total number of regions.

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