PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, May 1980

THE SPECTRUM OF THE LAPLACIAN OF KÄHLER MANIFOLDS

BANG-YEN CHEN AND LIEVEN VANHECKE

ABSTRACT. We strengthen some results on the spectrum of the Laplacian for 0- and 1-forms [8], [9] on Kähler manifolds and give some new results for the 2-forms.

1. Introduction. Let (M, J, g) be an n-dimensional Kähler manifold (all manifolds are assumed to be compact, connected and of complex dimension $n > 1$) with complex structure J and Kähler metric g. By Δ we denote the Laplacian acting on p-forms on M. Then we have the spectrum for each p,

$$\operatorname{Spec}^p(M, g) = \{0 \geqslant \lambda_{1,p} \geqslant \lambda_{2,p} \geqslant \cdots \downarrow -\infty\},$$

where each eigenvalue is repeated as many times as its multiplicity indicates. It is well known that $\operatorname{Spec}^p(M, g) = \operatorname{Spec}^{2n-p}(M, g)$ and, immediate from Hodge theory, that $0 \in \operatorname{Spec}^p(M, g)$ if and only if the pth Betti number $\beta_p(M) \neq 0$ (and 0 has multiplicity $\beta_p \neq 0$).

We denote by $(\mathbf{C}P^n, J_0, g_0)$ the n-dimensional complex projective space with the standard complex structure J_0 and the Fubini-Study metric g_0 of constant holomorphic sectional curvature c. One of the most interesting problems on spectrum is as follows: "Let (M, J, g) be a Kähler manifold with $\operatorname{Spec}^p(M, g) = \operatorname{Spec}^p(\mathbf{C}P^n, g_0)$ for a fixed p. Is it true that (M, J, g) is holomorphically isometric to $(\mathbf{C}P^n, J_0, g_0)$?"

In [1], [5] it is proved that if (M, J) is $(\mathbf{C}P^n, J_0)$, then the answer to the problem is affirmative for $p = 0$. In [8], [9] Tanno has proved that the answer is affirmative if $p = 0$ and $n \leqslant 6$ or if $p = 1$ and $8 \leqslant n \leqslant 51$.

The main purpose of this note is to prove the following.

THEOREM 1. *Let (M, J, g) be a Kähler manifold with* $\operatorname{Spec}^2(M, g) = \operatorname{Spec}^2(\mathbf{C}P^n, g_0)$. *If $n \neq 8$, then (M, J, g) is holomorphically isometric to* $(\mathbf{C}P^n, J_0, g_0)$.

The method we use in the proof is different from the previous authors since the second cohomology group $H^2(M, \mathbf{R})$ and the generalized Chern classes $c_1\omega^{n-1}$ and $c_1^2\omega^{n-2}$ play very important roles in the proof.

2. Preliminaries. Let M be a Kähler manifold of complex dimension n. If $(\theta^1, \ldots, \theta^n)$ form a local field of unitary coframes, the Kähler metric g and the fundamental 2-form Φ are given respectively by $g = \frac{1}{2}\sum(\theta^i \otimes \bar{\theta}^i + \bar{\theta}^i \otimes \theta^i)$ and $\Phi = \frac{1}{2}\sqrt{-1}\,\sum \theta^i \wedge \bar{\theta}^i$. Let $\Omega^i_j = \sum R^i_{jk\bar{l}}\theta^k \wedge \bar{\theta}^l$ be the curvature form of M. Then the curvature tensor R is the tensor field with local components $R^i_{jk\bar{l}}$. The Ricci tensor

Received by the editors December 5, 1978 and, in revised form, January 30, 1979.
AMS (MOS) subject classifications (1970). Primary 53C55; Secondary 58G99.
Key words and phrases. Spectrum, Laplacian, Kähler manifolds.

S and the scalar curvature ρ are given respectively by $S = \frac{1}{2}\Sigma(R_{i\bar{j}}\theta^i \wedge \bar{\theta}^j + \bar{R}_{i\bar{j}}\bar{\theta}^i \wedge \theta^j)$ and $\rho = 2\Sigma R_{i\bar{i}}$ where $R_{i\bar{j}} = 2\Sigma R^k_{ik\bar{j}}$. We denote by $\|R\|$ and $\|S\|$ the length of R and S respectively.

We define the kth scalar curvature ρ_k by

$$\det(\delta_{ij} + tR_{i\bar{j}}) = \sum_{k=0}^{n} \binom{n}{k} \rho_k t^k. \tag{2.1}$$

It is clear that $\rho_0 = 1$, $\rho_1 = \rho/2n$,

$$\rho_2 = \frac{1}{4n(n-1)}(\rho^2 - 2\|S\|^2), \ldots, \rho_n = \det(R_{i\bar{j}}).$$

Let ω be the cohomology class represented by the fundamental form Φ. Then ω is called the fundamental class and the last de Rham cohomology group $H^{2n}(M, \mathbf{R})$ is generated by ω^n. Let c_1 be the first Chern class of M. We put

$$\omega^{n-k}c_1^k = \alpha_k \omega^n, \qquad k = 0, 1, \ldots, n. \tag{2.2}$$

Then, the α_k are real numbers. We state the following lemma for later use [3] (see [6] for the cohomologically Einstein case).

LEMMA 1. *Let M be a Kähler manifold. Then we have*

$$\int_M \rho_k * 1 = (2\pi)^k \alpha_k \int_M * 1. \tag{2.3}$$

The Minakshisundaram-Pleijel-Gaffney formula for $\mathrm{Spec}^p(M, g)$ is given by

$$\sum_{k=0}^{\infty} e^{\lambda_{k,p} t} \underset{t\to 0}{\sim} \frac{1}{(4\pi t)^n} \sum_{i=0}^{\infty} a_{i,p} t^i$$

where

$$a_{0,p} = \binom{2n}{p} \int_M * 1, \tag{2.4}$$

$$a_{1,p} = \left\{\frac{1}{6}\binom{2n}{p} - \binom{2n-2}{p-1}\right\} \int_M \rho * 1, \tag{2.5}$$

$$a_{2,p} = \int_M \{c_1(2n,p)\rho^2 + c_2(2n,p)\|S\|^2 + c_3(2n,p)\|R\|^2\} * 1 \tag{2.6}$$

and

$$c_1(2n,p) = \frac{1}{72}\binom{2n}{p} - \frac{1}{6}\binom{2n-2}{p-1} + \frac{1}{2}\binom{2n-4}{p-2}, \tag{2.7}$$

$$c_2(2n,p) = -\frac{1}{180}\binom{2n}{p} + \frac{1}{2}\binom{2n-2}{p-1} - 2\binom{2n-4}{p-2}, \tag{2.8}$$

$$c_3(2n,p) = \frac{1}{180}\binom{2n}{p} - \frac{1}{12}\binom{2n-2}{p-1} + \frac{1}{2}\binom{2n-4}{p-2}. \tag{2.9}$$

The coefficients $a_{1,p}$ and $a_{2,p}$ are determined by Patodi [7].

3. Proof of Theorem 1. If $p = 2$, the first three coefficients in the asymptotic expansion are given by

$$a_{0,2} = n(2n-1)\int_M * 1, \tag{3.1}$$

$$a_{1,2} = \frac{2n^2 - 13n + 12}{6} \int_M \rho * 1, \tag{3.2}$$

$$a_{2,2} = \frac{1}{360} \int_M \{5(2n^2 - 25n + 60)\rho^2 - 2(2n^2 - 181n + 540)\|S\|^2 + 2(2n^2 - 31n + 120)\|R\|^2\} * 1. \tag{3.3}$$

Now, assume that $\mathrm{Spec}^2(M, g) = \mathrm{Spec}^2(\mathbf{C}P^n, g_0)$. Then we have

$$\dim_{\mathbf{C}} M = n, \qquad \dim H^2(M, \mathbf{R}) = \dim H^2(\mathbf{C}P^n, \mathbf{R}) = 1, \tag{3.4}$$

$$\int_M * 1 = \int_{\mathbf{C}P^n} * 1, \qquad \int_M \rho * 1 = \int_{\mathbf{C}P^n} \rho' * 1, \tag{3.5}$$

$$a_{2,2} = a'_{2,2}, \tag{3.6}$$

where ρ', $a'_{2,2}, \ldots$ etc. are the corresponding quantities for $(\mathbf{C}P^n, J_0, g_0)$.

From Lemma 1, we have

$$\int_M \rho * 1 = 4n\pi\alpha_1 \int_M * 1, \tag{3.7}$$

$$\int_M (\rho^2 - 2\|S\|^2) * 1 = 16n(n-1)\pi^2\alpha_2 \int_M * 1. \tag{3.8}$$

On the other hand, since the second Betti number of M is 1, $c_1 = a\omega$ for some real number a. Hence we have $\alpha_2 = \alpha_1^2$. Consequently, (3.7) and (3.8) imply

$$(n-1)\left(\int_M \rho * 1\right)^2 = n\left(\int_M * 1\right)\int_M (\rho^2 - 2\|S\|^2) * 1. \tag{3.9}$$

Since $(\mathbf{C}P^n, J_0, g_0)$ has constant holomorphic sectional curvature c, we have

$$\rho' = n(n+1)c, \qquad \|S'\|^2 = n(n+1)^2c^2/2, \qquad \|R'\|^2 = 2n(n+1)c^2. \tag{3.10}$$

From (3.5), (3.7) and (3.10) we find $\alpha_1 = (n+1)c/4\pi$. Using (3.3), (3.5) and (3.10) we have

$$a'_{2,2} = \frac{1}{360} n(n+1)(10n^4 - 117n^3 + 362n^2 - 183n - 60)c^2 \int_M * 1$$

and so

$$a'_{2,2} = \frac{16n\pi^2}{360(n+1)}(10n^4 - 117n^3 + 362n^2 - 183n - 60)\alpha_1^2 \int_M * 1.$$

Hence (3.7) implies

$$a'_{2,2} = \frac{1}{360n(n+1)}(10n^4 - 117n^3 + 362n^2 - 183n - 60)\frac{(\int_M \rho * 1)^2}{\int_M * 1}$$

and using (3.9) we obtain

$$a'_{2,2} = \frac{1}{360(n^2-1)}(10n^4 - 117n^3 + 362n^2 - 183n - 60)\int_M (\rho^2 - \|S\|^2) * 1.$$

Then from (3.6) we find

$$\begin{aligned}\int_M \{(n^2-1)(2n^2 - 31n + 120)\|R\|^2 \\ +4(2n^4 + 16n^3 - 44n^2 - 91n + 120)\|S\|^2 \\ - 2(2n^3 + 18n^2 - 77n + 60)\rho^2\} * 1 = 0, \quad (3.11)\end{aligned}$$

which is equivalent to

$$\begin{aligned}(n^2-1)(n-8)(2n-15)\int_M \left\{\|R\|^2 - \frac{2}{n(n+1)}\rho^2\right\} * 1 \\ +4(2n^4 + 16n^3 - 44n^2 - 91n + 120)\int_M \left\{\|S\|^2 - \frac{\rho^2}{2n}\right\} * 1 = 0. \quad (3.12)\end{aligned}$$

Since $\|R\|^2 \geqslant (2/n(n+1))\rho^2$ and $\|S\|^2 \geqslant \rho^2/2n$, equation (3.12) gives $\|R\|^2 = (2/n(n+1))\rho^2$ and $\|S\|^2 = \rho^2/2n$ if $n \neq 8$, from which we conclude that (M, J, g) has constant holomorphic sectional curvature, say $\bar{c}$. From (3.5) it follows then that $c = \bar{c}$. Thus, (M, J, g) is holomorphically isometric to $(\mathbf{C}P^n, J_0, g_0)$.

4. Remarks.

Remark 1. A Kähler manifold (M, J, g) is said to be *cohomologically Einsteinian* if $c_1 = a\omega$ for some real number a. By a similar argument as in the proof of Theorem 1, we may prove the following.

Theorem 2. *If (M, J, g) is a cohomological Einstein Kähler manifold and*

$$\mathrm{Spec}^0(M, g) = \mathrm{Spec}^0(\mathbf{C}P^n, g_0),$$

then (M, J, g) is holomorphically isometric to $(\mathbf{C}P^n, J_0, g_0)$.

Theorem 3. *If (M, J, g) is a cohomological Einstein Kähler manifold and*

$$\mathrm{Spec}^1(M, g) = \mathrm{Spec}^1(\mathbf{C}P^n, g_0), \qquad n > 7,$$

then (M, J, g) is holomorphically isometric to $(\mathbf{C}P^n, J_0, g_0)$.

Remark 2. If $H^2(M, \mathbf{R}) = \mathbf{R}$, then M is automatically cohomologically Einsteinian.

Corollary 4. *If (M, J, g) is a Kähler manifold with $H^2(M, \mathbf{R}) = \mathbf{R}$ and*

$$\mathrm{Spec}^0(M, g) = \mathrm{Spec}^0(\mathbf{C}P^n, g_0) \quad \big(\textit{or } \mathrm{Spec}^1(M, g) = \mathrm{Spec}^1(\mathbf{C}P^n, g_0) \textit{ and } n > 7\big),$$

then (M, J, g) is holomorphically isometric to $(\mathbf{C}P^n, J_0, g_0)$.

Remark 3. All complete intersections of dimension $n \geqslant 3$, all complete Kähler manifolds of positive curvature or of holomorphic pinching greater than $\frac{1}{2}$ **[2]** and all complete Kähler manifolds of positive holomorphic bisectional curvature **[4]** satisfy $H^2(M, \mathbf{R}) = \mathbf{R}$.

REMARK 4. From the proofs of Theorems 2 and 3 we may also see that we may replace the condition of "cohomologically Einsteinian" by the weaker condition "$\alpha_2 > \alpha_1^2$".

REFERENCES

1. M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété riemannienne*, Lecture Notes in Math., vol. 194, Springer-Verlag, Berlin and New York, 1971.

2. R. L. Bishop and S. I. Goldberg, *On the topology of positively curved Kaehler manifolds*. II, Tôhoku Math. J. **17** (1965), 310–318.

3. B. Y. Chen, *Characterizations of Einstein Kaehler manifolds and applications*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **61** (1976), 592–595.

4. S. I. Goldberg and S. Kobayashi, *Holomorphic bisectional curvature*, J. Differential Geometry **1** (1967), 225–233.

5. A. Lascoux and M. Berger, *Variétés Kählériennes compactes*, Lecture Notes in Math., vol. 154, Springer-Verlag, Berlin and New York, 1970.

6. K. Ogiue, *Generalized scalar curvatures of cohomological Einstein Kaehler manifolds*, J. Differential Geometry **10** (1975), 201–205.

7. V. K. Patodi, *Curvature and the fundamental solution of the heat operator*, J. Indian Math. Soc. **34** (1970), 269–285.

8. S. Tanno, *Eigenvalues of the Laplacian of Riemannian manifolds*, Tôhoku Math. J. **25** (1973), 391–403.

9. ______, *The spectrum of the Laplacian for 1-forms*, Proc. Amer. Math. Soc. **45** (1974), 125–129.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824

DEPARTMENT OF MATHEMATICS, KATHOLIEKE UNIVERSITEIT LEUVEN, CELESTIJNENLAAN 200B, B-3030 LEUVEN, BELGIUM