ISOMETRIC IMMERSIONS OF COMPLETE RIEMANNIAN MANIFOLDS INTO EUCLIDEAN SPACE

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ABSTRACT. Let $M$ be a complete Riemannian manifold of dimension $n$, with scalar curvature bounded from below. If the isometric immersion of $M$ into euclidean space of dimension $n + q$, $q < n - 1$, is included in a ball of radius $\lambda$, then the sectional curvature $K$ of $M$ satisfies $\limsup K > \lambda^{-2}$. The special case where $M$ is compact is due to Jacobowitz.

Generalizing results by Tompkins, Chern and Kuiper, and Otsuki, Jacobowitz proved that a compact $n$-dimensional Riemannian manifold whose sectional curvatures are everywhere less than constant $\lambda^{-2}$ cannot be isometrically immersed into euclidean space of dimension $2n - 1$ so as to be contained in a ball of radius $\lambda$ (see [1] and the references therein). In this note we shall prove a quantitative result concerning isometric immersions, which includes Jacobowitz’s theorem as a special case.

The proof of our result will consist in a simple application of a theorem by Omori [3], which we now formulate.

Let $M$ be a complete Riemannian manifold with sectional curvature bounded from below; consider a smooth function $f: M \to \mathbb{R}$ with $\sup f < \infty$. For any $\epsilon > 0$ there exists a point $p \in M$ where $||\text{grad } f|| < \epsilon$ and $\nabla^2 f(X, X) < \epsilon$ for all unit vectors $X \in T_pM$. By $\nabla^2 f$ we mean the Hessian form of $f$, defined by $\nabla^2 f(X, Y) = \langle \nabla_X \text{grad } f, Y \rangle$.

**Theorem 1.** Let $M$ be a complete $n$-dimensional Riemannian manifold with scalar curvature $R$ bounded from below. Assume that there exists an isometric immersion $\varphi$ of $M$ into euclidean space of dimension $n + q$, $q < n - 1$, so that $\varphi(M)$ is included in a ball of radius $\lambda$. Then $\limsup K > \lambda^{-2}$, where $K$ is the sectional curvature of $M$.

**Corollary.** A complete two-dimensional Riemannian manifold, immersed isometrically into euclidean three-space, and whose Gaussian curvature $K$ satisfies $-\infty < -a^2 < K < 0$, is extrinsically unbounded.

**Proof of the Theorem.** If $n = 2$ then $R = 2K$ and we have $\inf K > -\infty$. If $n > 2$ and $\inf K = -\infty$, then $\inf R > -\infty$ easily implies $\sup K = +\infty$ and the theorem follows. We may therefore assume $\inf K > -\infty$.

We shall apply Omori’s theorem to the “distance” function $F = \langle \varphi, \varphi \rangle/2$; $\varphi$ is considered here as tangent vector in euclidean space $E^{n+q}$. By assumption, we have

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\( \| \varphi \| < \lambda \) and \( f < \lambda^2/2 \), taking the origin to be the center of the ball wherein \( \varphi(M) \) lies. Therefore, to any natural number \( m \), there exists a point \( p_m \in M \) where \( \nabla^2 f(X, X) < 1/m \) for all \( X \in T_{p_m}M \) with \( \| X \| = 1 \). In order to compute the Hessian of \( f \), we identify every tangent vector \( X \) with \( \varphi_*(X) \) and obtain
\[
\nabla^2 \varphi = X,
\]
where \( \nabla^2 \) denotes the connection of \( E^{n+q} \). Now using this and the Gauss formula, we compute easily \( \nabla^2 f(X, Y) = \langle \nabla X, Y \rangle + \langle L(X, Y), \varphi \rangle \), where \( L \) stands for the second fundamental form of the immersion. Thus at \( p_m \) and for every nonzero \( X \in T_{p_m}M \) we have \( 1 + \langle L(X, X), \varphi \rangle \cdot \| X \|^{-2} < m^{-1} \), hence
\[
\lambda^{-1}(1 - m^{-1}) < \| L(X, X) \| \cdot \| X \|^{-2}.
\]
From (*) we conclude that, at \( p_m \in M \), we have \( L(X, X) \neq 0 \) for \( X \neq 0 \). Now we use, as in [1], a well-known algebraic lemma [2, p. 28]. Let \( L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^q \) be symmetric, bilinear and satisfy \( L(X, X) \neq 0 \) for \( X \neq 0 \); if \( q < n - 1 \), there exist linearly independent \( X, Y \) so that \( L(X, Y) = 0 \) and \( L(X, X) = L(Y, Y) \). We pick two such vectors \( X, Y \) in \( T_{p_m}M \), apply (*) and obtain
\[
\lambda^{-2}(1 - m^{-1})^2 < \| L(X, X) \| \cdot \| L(Y, Y) \| \cdot \| X \|^{-2} \cdot \| Y \|^{-2} < (\langle L(X, X), L(Y, Y) \rangle - \| L(X, Y) \|^2) \cdot (\| X \|^2 \| Y \|^2 - \langle X, Y \rangle^2)^{-1}.
\]
By the Gauss equation, the rightmost term in these inequalities is the sectional curvature of \( M \) at \( p_m \) for the plane spanned by \( X \) and \( Y \). Now letting \( m \) go to infinity, we deduce \( \lambda^{-2} \leq \lim \sup_{p \in M} K(X \wedge Y) \) and thus prove the theorem.

It is noteworthy that the above proof includes a generalization of the following well-known result. If a compact hypersurface \( M \) in \( E^N \) is contained in a ball of radius \( \lambda \), then there exists a point on \( M \) where all the normal curvatures are in absolute value not less than \( \lambda^{-1} \). For a submanifold \( M \) of \( E^N \), of arbitrary codimension, we define the absolute normal curvature at a point \( p \in M \) and in the direction \( X \in T_pM, \| X \| = 1 \), to be \( \| L(X, X) \| \). Let
\[
C(p) = \min \{ \| L(X, X) \| / X \in T_pM \text{ and } \| X \| = 1 \}.
\]

**THEOREM 2.** Let \( M \) be a complete submanifold of \( E^N \) with sectional curvature bounded from below. If \( M \) is contained in a ball of radius \( \lambda \), then \( \lim \sup_{p \in M} C(p) > \lambda^{-1} \).

**PROOF.** Apply Omori's theorem as in Theorem 1 to \( \langle \varphi, \varphi \rangle/2 \). From inequality (*) we immediately obtain the conclusion.

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**REFERENCES**


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