ON THE ITERATIONS OF DIFFEOMORPHISMS WITHOUT C°-Ω-EXPLOSIONS: AN EXAMPLE

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ABSTRACT. In this note we construct a diffeomorphism \( f \) such that \( f \) has no C°-Ω-explosion but \( f^2 \) has Ω-explosion.

The purpose of this note is to give an example of a diffeomorphism \( f \) on a 2-sphere \( S^2 \) such that \( f \) has no C°-Ω-explosion but \( f^2 \) has Ω-explosion. As noted below, this is accomplished by finding a diffeomorphism \( f \) without C°-Ω-explosion for which \( Ω(f) ≠ Ω(f^2) \).

Before proceeding to construct the example, we shall make a few observations on the iterations of diffeomorphisms without C°-Ω-explosions:

A point \( x \in S^2 \) is said to be a chain recurrent point of \( f \) if for any \( ε > 0 \) there exists a sequence \( \{x_0, \ldots, x_n\} \) of points on \( S^2 \) with \( x_0 = x_n = x \) and \( d(f(x_i), x_{i+1}) < ε \) where \( d \) is a metric on \( S^2 \) (cf. [1]). We denote the sets of chain recurrent points and nonwandering points of \( f \) by \( θl(f) \) and \( θf(f) \) respectively. By definitions, it follows that \( Ω(f^m) \subset Ω(f) \subset θf(f) = θl(f^m), m ≠ 0 \). In [2, Theorem 3.11] M. Shub showed that \( f \) has no C°-Ω-explosion if and only if \( θl(f) = θf(f) \). From the above facts, we have

**Proposition.** A diffeomorphism \( f^m \) has no C°-Ω-explosion if and only if \( f \) has no C°-Ω-explosion and \( Ω(f) = Ω(f^m) \).

Hence for our purpose, it is sufficient to construct a diffeomorphism \( f \) on \( S^2 \) such that \( f \) has no C°-Ω-explosion and \( Ω(f) ≠ Ω(f^2) \), i.e., \( θl(f) = Ω(f) ≠ Ω(f^2) \).

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**The construction.** At first we take a diffeomorphism \( f' \) on \( S^2 \) to be the time one map of the flow \( ψ_t \) on \( S^2 \) as pictured in Figures 1 and 2. Here Figures 1 and 2 show the flow \( ψ_t \) on the upper and the lower hemispheres, \( H^+ \) and \( H^- \) respectively, and \( S^1 \) is the equator, \( 0 \) and \( 0' \) are the north and the south poles respectively, and \( A^+ \) is the antipodal point of \( A \).

For \( f' \), there are two fixed points \( A \) and \( A' \), two fixed sources \( 0 \) and \( 0' \), and clearly \( S^1 \) is an invariant set. Furthermore let \( f \) satisfy that \( |θ(f'(x))| < |θ(x)| \) for any \( x \in S^2 - (S^1 ∪ 0 ∪ 0') \) where \( θ(x) \) is the latitude of \( x \) \((-π/2 < θ(x) < π/2)\) and \(|(\ldots)|\) is the absolute value of \((\ldots)\).
Now we construct a diffeomorphism \( f \) on \( S^2 \) as follows; \( f = \rho \circ f' \) where \( \rho \) is the map such that \( \rho(x) \) is the antipodal point of \( x \) for any \( x \in S^2 \). Note that \( A, A' \) are periodic points, \( 0, 0' \) are periodic sources of period 2 of \( f \), and that \( f \) satisfies

\[
|\theta(f(x))| < |\theta(x)| \quad \text{for any } x \in S^2 - (S^1 \cup 0 \cup 0'). \tag{\ast}
\]

**Lemma.** \( \mathcal{R}(f) = \Omega(f) \neq \Omega(f^2) \).

**Proof.** We first show that \( \mathcal{R}(f) = 0 \cup 0' \cup S^1 \). Clearly \( 0 \cup 0' \cup S^1 \subset \mathcal{R}(f) \). Hence it suffices to show that \( x \notin \mathcal{R}(f) \) for \( x \in S^2 - (0 \cup 0' \cup S^1) \). Let \( \theta = |\theta(x)| \) and \( B = \{ y \in S^2 : |\theta(y)| < \theta \} \). Then by (\ast), \( |\theta(f(y))| < \theta \) for any \( y \in B \).

Since \( B \) is compact, there exists \( \epsilon > 0 \) such that \( |\theta(y')| < \theta \) for any \( y' \in U_\epsilon(f(B)) \) (\( U_\epsilon(\ldots) \) is an \( \epsilon \)-neighborhood of \( \ldots \)). Let \( \{x_0, \ldots, x_m\} \) be a sequence of points on \( S^2 \) with \( x_0 = x \), \( d(f(x_i), x_{i+1}) < \epsilon \). Then \( x_1 \in U_\epsilon(f(B)) \subset B \) since \( x_0 = x \in B \). By induction, \( x_m \in U_\epsilon(f(B)) \) so that \( |\theta(x_m)| < \theta = |\theta(x)| \). Therefore there exists no sequence \( \{x_0, \ldots, x_n\} \) of points on \( S^2 \) with \( x_0 = x_n = x \) and \( d(f(x_i), x_{i+1}) < \epsilon \). Hence \( x \notin \mathcal{R}(f) \) and \( \mathcal{R}(f) = 0 \cup 0' \cup S^1 \).

We next show that \( \mathcal{R}(f) = \Omega(f) \neq \Omega(f^2) \). Clearly \( 0, 0', A, A' \in \Omega(f) \in \Omega(f^2) \).

Let \( p \in S^1 - (A \cup A') \), \( V \) be a neighborhood of \( p \) and \( V^+ = V \cap (H^+ - S^1) \).

Without loss of generality, we may assume that \( f^{2n}(p) \to A' \) as \( n \to \infty \). Then
$f^{2n}(V^+)$ is pressed toward $S^1$ as in Figure 3. Therefore for a sufficiently large $n$, $f^{2n+1}(V) \cap V = f^{2n+1}(V^+) \cap V \neq \emptyset$. Hence $p \in \Omega(f)$ and $\mathcal{R}(f) = \Omega(f)$.

On the other hand, $(f^2)^n(V) \cap V = f^{2n}(V) \cap V = \emptyset$. Hence $p \not\in \Omega(f^2)$. Therefore $\mathcal{R}(f) = \Omega(f) \neq \Omega(f^2)$.

Hence $f$ has no $C^0$-$\Omega$-explosion but $f^2$ has $\Omega$-explosion.

REFERENCES


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