ON ODD-PRIMARY COMPONENTS OF LIE GROUPS

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Abstract. The transfer map \( t: \pi^*(P_{\infty}C^+) \to \pi^*(S^0) \) is represented by an element \( \tau \in \pi_{-1}^*(P_{\infty}C^+) \). We compute the Adams-e-invariant of \( \tau \) and use this and the splitting of the \( p \)-localization of \( S^1 \setminus P_{\infty}C \) into a wedge of \((p - 1)\) spaces to prove that for a prime \( p > 5 \) the \( p \)-component of the element \([G, \xi]\) defined by a compact Lie group \( G \) in \( \pi^*_p \) is zero in the known part of stable homotopy.

1. The e-invariant of the transfer map. The easiest definition of the transfer \( t_{s1} : \pi_{-1}^*(P_{\infty}C^+) \to \pi_{-1}^*(S^0) \) is in terms of framed bordism: An element in \( \pi_{-1}^*(P_{\infty}C^+) \) is given by a framed manifold \((M, \Phi)\) together with an \( S^1 \)-principal bundle \( \xi \). The total space of \( \xi \) together with the canonical framing constructed from \( \Phi \) represents \( t_{s1}(M, \Phi) \). On finite skeletons we can represent \( t_{s1} \) by stable maps, which fit together to give an element \( \tau \) in \( \pi_{-1}^*(P_{\infty}C^+) \). The map \( \tau \) induces the transfer \( t_{s1} \) in stable cohomotopy and we have \( \tau = t_{s1}(1) \). Because we are always working in a fixed dimension in homology, we can avoid limit discussions by restricting to a finite skeleton. Let \( \beta \) be the Bockstein map \( \beta : \pi_{-2}^*(P_{\infty}C^+; Q/Z) \to \pi_{-1}^*(P_{\infty}C^+) \). Because \( \pi_{-1}^*(P_{\infty}C^+) \) is finite, we can find \( \bar{\tau} \) with \( \beta(\bar{\tau}) = \tau \). Then the e-invariant of \( \tau \) is given by \( e(\bar{\tau}) \), or equivalently by \( \bar{\tau} \in \text{Hom}(K_0(P_{\infty}C), K_0(*; Q/Z)) \cong K^0(P_{\infty}C; Q/Z) \) (see [5]). To compute \( \bar{\tau} \), we write \( t_{s1} \) as a composition of two transfer maps:

\[
\pi_{-1}^*(P_{\infty}C^+) \xrightarrow{t_0} \pi_{-1}^*(BZm^+) \xrightarrow{t_m} \pi_{-1}^*(S^0).
\]

The element \( t_{s1}(1) \in \pi_{-1}^*(BZm^+) \) is not in the image of \( \beta \), but \( t_{s1}(1) - m \) is. So \( t_{s1}(1) \) and \( t_m \) differ by \( m \cdot t_m(1) \).

Lemma 1.1. For \( n \) fixed, there is an \( m \) such that \( m \cdot t_m(1) \in \pi_{-1}^*(P_{n}C) \) is zero.

Proof. Let \( L \) be the universal line bundle over \( P_{n}C \). Then the sphere bundle of \( L^m \) is the \((2n + 1)\)-skeleton of \( BZ_m \). There exists a number \( m \) such that \( J(L^m) = 0 \) in \( J(P_{n}C) \), so \( L^m \) is orientable for \( t_m^* \) and we have an exact Gysin sequence

\[
\pi_{-1}^*(P_{n}C^+) \xrightarrow{t_m^*} \pi_{-1}^*((BZm^+)^{2n+1}) \xrightarrow{t_m} \pi_{-1}^*(P_{n}C^+) \to .
\]

Let \( pr: P_{n}C \to * \) denote the projection; then \( t_m^*(1) = t_m \circ pr^*(1) \) factors over \( t_m \circ t_m^* \), so it must be zero.

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So for elements of filtration less than \( n \) and \( m \) large \( \bar{\tau}_* \) is given by the composition:

\[
K_0(P_\infty C) \xrightarrow{t_m} K_1(B\mathbb{Z}_m) \xrightarrow{\beta} \bar{K}_2(B\mathbb{Z}_m; \mathbb{Q}/\mathbb{Z}) \xrightarrow{t_m} K_2(\ast; \bar{\mathbb{Q}}/\mathbb{Z}).
\]

**Proposition 1.2.** The element \((t_{Z_m} \circ \beta^{-1} \circ t_m) \in \text{Hom}(K_0(P_\infty C); \mathbb{Q}/\mathbb{Z})\) is given by the Kronecker product with \( \text{red}(1/(1 - L) - m/(1 - L^m)) \in K^{-2}(P_\infty C; \mathbb{Q}/\mathbb{Z})\), where \( \text{red}: K^*(P_\infty C; \mathbb{Q}) \rightarrow K^*(P_\infty C; \mathbb{Q}/\mathbb{Z}) \) is the reduction mod \( \mathbb{Z} \) and \( L \) the universal line bundle.

**Proof.** We have \( t_{Z_m} \beta^{-1} t_m(z) = \langle 1, t_{Z_m} \beta^{-1} \circ t_m(z) \rangle_K = \langle t_{Z_m}(1) - m, \beta^{-1} \circ t_m(z) \rangle \) where \( \langle , \rangle_K \) is the Kronecker product \( K^*(X) \times K_*(X; \mathbb{Q}/\mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z} \). It is well known that \( t_{Z_m}(1) \) is given by the regular representation of \( \mathbb{Z}_m \), that is by \( t_{Z_m}(1) = \sum_{i=0}^{m-1} \pi_i(L^i) \), where \( \pi: B\mathbb{Z}_m \rightarrow P_\infty C \) is the projection. To compute \( t_m \beta^{-1} \pi(L^i - 1) \), we observe that the transfer \( t_m \) composed with the Thom isomorphism \( \phi \) of the bundle \( L^m \) is the coboundary map \( \delta: K^1(S(L^m); \mathbb{Q}/\mathbb{Z}) \rightarrow K^0(D(L^m), S(L^m); \mathbb{Q}/\mathbb{Z}) \). This follows from [2], where it is proved that \( t_m \) is the Umkehr-map of \( \pi \) and an easy calculation with Poincaré duality. We therefore consider \( \delta \circ \beta^{-1} \circ \pi^* \). Let \( \text{red} \circ \delta \circ \beta^{-1} \) be the functional cohomology operation defined by the diagram

\[
\begin{array}{c}
\rightarrow K^1(S; \mathbb{Q}) \\
\downarrow \delta \\
K^0(D, S; \mathbb{Q}) \xrightarrow{\text{red}} K^0(D, S; \mathbb{Q}/\mathbb{Z}) \rightarrow K^1(D, S; \mathbb{Z})
\end{array}
\]

It is easy to see that this operation is up to a sign the same as the one defined by the diagram

\[
\begin{array}{c}
K^0(D, S) \\
\downarrow \delta \\
K^0(D; \mathbb{Q}) \rightarrow K^0(S; \mathbb{Q}) \rightarrow K^1(D, S)
\end{array}
\]

Thus \( \text{red}^{-1} \circ \delta \circ \beta^{-1}(z) = j^{*-1} \pi \pi^* \phi^* \). The element \((1 - L^m)/(1 - L) = \sum_{i=1}^{m-1} (L^{i-1} - 1) \) is invertible in \( K^0(P_\infty C; \mathbb{Q}) \); thus the element \((L^{i-1} - 1)/(1 - L^m) \) in \( K^0(P_\infty C; \mathbb{Q}) \) is well defined.

Because \( j^{*-1} \circ \phi \) is the cup product with the Euler class \( e(L^m) = 1 - L^m \) we have \( \phi^{-1} \circ j^{*-1} \circ \phi(L^{i-1}) = \text{red}((L^{i-1} - 1)/(1 - L^m)) \) and so \( t_m \beta^{-1} \pi^*(L^{i-1}) = \text{red}((L^{i-1} - 1)/(1 - L^m)) \). This gives

\[
t_m \beta^{-1}(t_{Z_m}(1) - m) = \text{red} \left( \sum_{i=1}^{m-1} (L^{i-1} - 1)/(1 - L^m) \right)
= \text{red} \left( \sum_{i=0}^{m-1} L^i - m \right) / (1 - L^m)
= \text{red} \left( (L^m - 1)/(L - 1) - m/(1 - L^m) \right)
= \text{red}(1/(1 - L) - m/(1 - L^m)).
\]
Theorem 1.3. The $e$-invariant of $\tau$ is given by the power series\[1/x - 1/log(x + 1)\]in $K^{-2}(P_{\infty C}; Q/Z)$ where $x = L - 1$.

Proof. Given an element $w$ in $\text{im}(K_0(P_\infty C) \to K_0(P_{\infty C}))$ we have\[\langle 1/x - m/(1 - L^m), w \rangle_K = \langle \text{ch}(x)^{-1} - m \cdot \text{ch}(1 - L^m)^{-1}, \text{ch}(w) \rangle_H\]
where $z = c_i(L)$. The power series of $z^{-1} - m/(1 - e^{mu})$ shows that for $m$ large the last product becomes integral; so
\[\langle (1 - L)^{-1} - m/(1 - L^m), w \rangle_K \equiv \langle x^{-1} - 1/log(x + 1), w \rangle_K \mod Z.\]

The slant product with the element $w = x^{-1} - 1/log(x + 1) \in K^{-2}(P_{\infty C}; Q/Z)$ defines a map $\hat{t}: K_0(BT^n) \to K_2(BT_{n-1}; Q/Z)$. Because the transfer of the fibre bundle $BT_{n-1} \times ES^1 \to BT^n$ is induced by the stable map id $\land \tau$ we have from Theorem 1.3:

Corollary 1.4. The composition\[\pi^{i_2}_{2m}(BT_{n+1}) \to \pi^{i_2}_{2m+i}(BT^{n+1}) \to K_0(BT^n; Q/Z)/\text{im} H_{2m}(BT^n; Q)\]
is given by $\hat{t} \circ h$, where $h: \pi^{i_2}_{2m}(BT^{n+1}) \to K_0(BT^n; Q)$ is the Hurewicz map.

For applications of Corollary 1.4 see [5].

Let $p$ be an odd prime. Then some suspension of the $p$-localization of $P_\infty C$ splits into a wedge of $(p - 1)$ spaces\[S' \land P_n C_{(p)} \simeq X_1 \lor X_2 \lor \cdots \lor X_{p-1}\]
where $X_i$ has only cells in dimensions $2i + 2i(p - 1) + r$ (for a proof see [7]). Therefore the stable map $\tau \in \pi^{i_2}_{2m}(P_{\infty C}^p)$ decomposes into a sum of $\tau_i \in \pi^*_i(X_i)$.

Proposition 1.5. Let $p$ be an odd prime, then $e_*(\tau_i) = 0$ if $i \equiv -1 (p-1)$.

Proof. The class $x^{-1} - 1/log(x + 1)$ in $K^{-2}(P_{\infty C}; Q)$ is mapped under the Chern character into $(1 - e^{x^{-1}})^{-1} - z^{-1} = \Sigma_{i=0}^\infty (B_{i+1}/i + 1) \cdot z^{i}/i!$. So $e(\tau_i) = \text{red} \circ \text{ch}^{-1}(f_i)$ where $f_i : = \Sigma_{i=0}^\infty (B_{i+1}/i + 1) \cdot z^{i}/i!$ in $H^*(X_i; Q)$. Now the cannibalistic characteristic class $\rho^k: K^*(X) \to K^*(X) \otimes \mathbb{Z}[1/k]$ operates on the $2n$-sphere as multiplication by $(k^n - 1) \cdot B_n/n$ [1]. It is easy to see that $\psi^{k - 1}: K^*(X_i; Z_{(p)}) \to K^*(X_i; Z_{(p)})$ is an isomorphism for $k \equiv 0 (p)$ and $i \equiv -1 (p-1)$. So $\rho^k \circ (\psi^{k - 1})^{-1}(a)$ is a well-defined class in $K^{-1}(X_i; Z_{(p)})$. If $x$ is the class of $L - 1$ in $K^*(X_i; Z_{(p)})$, then $\chi \rho^{-k} \circ (\psi^{k - 1})^{-1}(x) = f_i$ so $e(\tau_i) \equiv 0 \mod Z_{(p)}$.

A basis of $H^2(BT^n; \mathbb{Z})$ defines a homeomorphism $g: BT^n \to P_\infty C \land \cdots \land P_\infty C$. Using $g$ we decompose the suspension of $BT^n$ into a wedge of smash products of $P_\infty C$ and so, after localization, into a wedge of smash products of the $X_i$.

By (2.1) of [5] and Proposition 1.5 we find that the transfer $t: \pi^{i_2}_{2m}(S^n)_{(p)} \to \pi^{i_2}_{2m}(S^0)_{(p)}$ is concentrated on the component $\pi^{i_2}_{2m}(X, X \land X, \land \cdots \land X)$ (n factors with $r \equiv -1 (p-1)$). That is to say, only on this component can $t$ raise the filtration associated to the BP-Adams spectral sequence by $n$. On all other components $t$ must raise the filtration at least by $n + 2 (p-1)$.
Given an element \((B, f) \in \pi_2^s(B^m)\) we can use the Hurewicz map \(h: \pi_2^s(B^m) \to H_2(B^m)\) to find out when \((B, f)\) has a component in \(\pi_2^s(X_r \wedge X_r \wedge X_r)\) up to higher filtration. Because the filtration increases in steps of \(2 (p - 1)\) on spaces like \(X_r \wedge X_r \wedge \cdots \wedge X_r\) we have:

**Corollary 1.6.** Given \(z \in \pi_2^s(B^m)\) with \(h(z) = 0\) in \(H_2(X_r \wedge \cdots \wedge X_r)\), then \(T(z) \in \pi_2^s(S^0)\) is at least of filtration \(n + 2 (p - 1)\).

2. Application to Lie groups. Let \(G\) be a compact Lie group of rank \(n\) with maximal torus \(T\). The left invariant framing \(\mathcal{F}\) of \(G\) induces a framing \(\mathcal{F}\) of \(G/T\). Together with the classifying map \(f\) of the bundle \(G \to G/T\) we get an element \([G/T, f, \mathcal{F}] \in \pi_*^s(B^m) \approx \pi_*^s(B^m)\). The image of this element under the transfer \(\pi_*^s(B^m) \to \pi_*^s(S^0)\) is the element \([G, \mathcal{F}]\) defined by the Lie group in \(\pi_*^s(S^0)\) [5].

Let \(z_1, \ldots, z_n\) be a basis of \(H^2(B^m; \mathbb{Z})\). Then the image of \([G/T, f, \mathcal{F}]\) under the Hurewicz map \(h: \pi_*^s(B^m) \to H_*^s(B^m)\) is determined by the Kronecker products

\[
c_k = \langle f^*(z_1^k \cup \cdots \cup z_n^k), [G/T] \rangle_H
\]

where \(2\Sigma_i k_i = \dim G/T\). Let \(x_1, \ldots, x_n\) be a basis of \(H^1(T^n; \mathbb{Z})\) and \(\tau: H^1(T^n; \mathbb{Z}) \to H^1(G/T; \mathbb{Z})\) the transgression map. We can choose \(x_1, \ldots, x_n\) such that \(\tau(x_i) = f^*(z_i)\). In the following we will identify \(H^1(T^n; \mathbb{Z})\) with the dual of the integer lattice of \(G\).

A set of \(n\) linearly independent elements in \(H^2(G/T; \mathbb{Z})\) defines a lattice \(\Gamma\) in \(H^2(G/T)\) and a torus bundle \(E \to G/T\). The total space \(E\) is then the quotient of \(G\) by a finite group \(H\) with order \(|H| = \text{index of } \Gamma\). (The manifold \(E\) is framed in a canonical way.) In considering the \(p\)-component only, we really do not need a basis of \(H^2(B^m; \mathbb{Z})\) but only one of \(H^2(G/T; \mathbb{Z})\) because the use of \(m \cdot z_i\) for \(m \in \mathbb{Z}\) means that we turn from \(G\) to the framed manifold \(\overline{G} = G/Z_m\). But if \(m \equiv 0 (p)\) then \(m \cdot [\overline{G}, \mathcal{F}]_{(p)} = [G, \mathcal{F}]_{(p)}\).

**Proposition 2.1.** Let \(p > 3\) be a prime and \(G\) a compact Lie group of rank \(n\). Then there exists a decomposition of \(SBT^m\) such that the component of \(h([G/T, f, \mathcal{F}])\) in \(H_*(X_r \wedge \cdots \wedge X_r)\) \((r = n - 1 (p - 1), n \text{ factors})\) is zero.

**Proof.** We only need to prove that for all primes \(p > 3\) and all compact Lie groups \(G\) there exists a basis \(z_1, \ldots, z_n\) such that the numbers \(c_{(k)}\) with all \(k_\ell \equiv -1 (p - 1)\) vanish. The general argument is as follows: First let \(G\) be simple. We look for classifying elements \(f^*(z_i) = \gamma_i\) such that there exist for each pair \((i, j), i \neq j\), an element \(w\) in the Weyl group of \(G\) which permutes \(\gamma_i\) and \(\gamma_j\), leaves all the others fixed and operates on the fundamental class \([G/T]\) as multiplication by \(-1\). Let \(z = \gamma_1^k \cup \cdots \cup \gamma_n^k\) with \(k_\ell \equiv -1 (p - 1)\). If \(\langle z, [G/T] \rangle \neq 0\) then all \(k_\ell\) must be different, for if \(k_\ell = k_j\) with \(i \neq j\), we have a \(w \in W(G)\) with \(w^*(\gamma_i) = \gamma_j\), that is \(w^*(z) = z\); but \(w_*[G/T] = - [G/T]\). By calculating the dimension of such \(z\) we then see that all the corresponding \(c_{(k)}\) must vanish. For a semisimple Lie
group we look for such elements in each simple component. We call such a set of classifying elements a \(*\)-basis.

The existence of a \(*\)-basis can be easily checked for the simply connected simple Lie groups:

1. $A_n$, $B_n$, $C_n$, $D_n$. We can use as a \(*\)-basis the elements denoted by $e_i$ in [4]. For the dimension argument to hold for $B_n$ and $C_n$ we must suppose $p > 3$ whereas $p > 2$ suffices for $A_n$ and $D_n$.

2. For $G_2$ see [3].

3. For $F_4$ we refer to [3]. It is an exercise to see that with respect to the given basis in [3] there are only the possibilities $(1, 5, 7, 11)$, $(1, 3, 9, 11)$, $(3, 5, 7, 9)$, $(1, 3, 7, 13)$, and $(1, 3, 5, 15)$ for exponent sequences $k_i$.

4. The cases $E_6$, $E_7$, $E_8$ can be treated using exercises 29 and 30 in Chapter 4 of [9].

Now let $G$ be semisimple. Then $G$ is the quotient of a product of simple Lie groups by a finite subgroup of the center of this product. Because we have $p > 3$ we only have to consider subgroups of the center of a product of $SU(n_j)$'s.

Let $\hat{G} = SU(n_1) \times \cdots \times SU(n_m)$, $H \subset \text{center}(\hat{G})$, $G = \hat{G}/H$ and set $\hat{G} = \hat{G}/\text{center}(\hat{G})$. We denote the dual of the integer lattice of a Lie group $G$ by $I_G^\ast$. To the covering $\hat{G} \to G$ there corresponds an inclusion of lattices $I_G^\ast \subset I_H^\ast$. Contained in $I_G^\ast$ is $I_H^\ast$. It suffices to find a \(*\)-basis for $I^\ast \otimes \mathbb{Z}(p)$ for all lattices $I^\ast$ between $I_H^\ast$ and $I_G^\ast$. The lattices $I_H^\ast$ and $I_G^\ast$ are product lattices. Using the notation of [4] we have $I_G^\ast = \prod_{j=1}^{m_j}\langle e_{i_1}^j, \ldots, e_{i_{m_j}}^j \rangle$ and $I_H^\ast = \prod_{j=1}^{m_j}\langle e_{i_1}^j - e_2^j, \ldots, e_{i_l}^j - e_{i_{m_j}}^j \rangle$ (we set $\sum_i e_i^j = 0$).

Let $F = I^\ast/I_H^\ast$ and $\pi^\ast : I^\ast \to F$ be the projection. We can suppose that $F$ is a $p$-group. Let $z_1, \ldots, z_j$ be preimages of the generators of the factors of $F$ under $\pi$. Because of $\pi(e_i^j) = \pi(e_i^k)$ we can write $z_j$ as a linear combination of the $e_i^j$: $z_j = \sum_{i=1}^{m_j} a_i^j \cdot e_i^j$. We have $n_j \cdot e_i^j \in I_H^\ast$. Furthermore we can choose the $z_j$ in such a way that $a_i^j = 0$ for $i < j$ and $n_j \cdot e_i^j / a_i^j > n_i \cdot e_i^j / a_i^j$ for $j > k$ and $k = 1, \ldots, f$. It is then clear that $I^\ast = I_H^\ast + \langle z_1, \ldots, z_j \rangle$.

We define the lattice $\Gamma$ to be generated by $e_1^1 + z_1 - e_2^1$, $e_1^1 + z_1 - e_3^1$, $e_1^1 + z_1 - e_4^1$, $e_1^1 + z_1 - e_5^1$, $e_2^1 + e_2 - e_3^1$, $e_2^1 + e_2 - e_4^1$, $e_2^1 + e_2 - e_5^1$, $e_3^1 + z_1 - e_4^1$, $e_3^1 + z_1 - e_5^1$, $e_4^1 + z_1 - e_5^1$. These elements form a \(*\)-basis for $\Gamma$. It is easy to see that $\Gamma \otimes \mathbb{Z}(p) = I^\ast \otimes \mathbb{Z}(p)$.

In the general case, where $G$ is not semisimple, it is clear that there is always at least one basis element of $H^1(T^n; \mathbb{Z})$ lying in the kernel of the transgression map. So there is no nonzero $c_{(k)}$ in which all classifying elements appear.

**Corollary 2.2.** Let $p > 3$ be a prime and $G$ a compact Lie group of rank $n$. Then the $p$-component of $[G, \mathcal{C}]$ in $\pi^\ast_*(S^0)$ is at least of filtration $n + 2(p - 1)$ where the filtration is associated to the Adams spectral sequence for $BP$.

The vanishing line of the $E^2$-term of the Adams spectral sequence for $BP$ (see [6]) shows that elements of high filtration cannot exist in low dimensions. This and (2.2) and some simple dimension arguments show that in the known part of stable homotopy—see for example [8]—we have $[G, \mathcal{C}_{k(p)}] = 0$ for $p > 3$. This also shows that $[G, \mathcal{C}_{k(p)}]$ for an exceptional Lie group $G$ can be nonzero only for $p = 2$ or 3.
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