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A CLASS OF 4-MANIFOLDS WHICH HAVE 2-SPINES[1]

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ABSTRACT. In this note we establish the following.

THEOREM. *Let M^4 denote a piecewise linear or smooth compact 4-manifold-with-boundary having a handle decomposition consisting of* 0-, 2- *and* 3-*handles. Then M^4 has a* 2-*dimensional spine if and only if $H_3(M^4) = 0$.*

The focal point of this article is the technique developed to obtain the above-stated result. Observing that M^4 has a 2-dimensional spine if and only if it has a handle presentation consisting of 0-, 1- and 2-handles, the trick is to construct a complementary 2-handle to each of the given 3-handles–without introducing new 3-handles in the process. This is accomplished as follows.

We view the given 0-handle as $B^3 \times I$. Note that we may assume the union of the 2- and 3-handles meets the 0-handle in the interior of $B^3 \times \{1\}$. Using the hypothesis $H_3(M^4) = 0$, to each of the given 3-handles we shall construct a complementary arc in $B^3 \times \{1\}$. To be more precise, each arc will be properly embedded in $(B^3 \times \{1\})$ − (the union of the 2-handles) and each arc will meet the union of the 3-handles in a single subarc. This subarc will be a B^1-factor of the attaching tube of the associated 3-handle. The desired complementary 2-handles will be regular neighborhoods of these arcs cross the interval in $B^3 \times I$.

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Definitions and Notation. We shall let $I = [0, 1]$, and B^n (int B^n) will be used to denote the closed (open) unit ball about the origin in Euclidean n-space.

We say the n-manifold M^n is obtained from the n-manifold W^n by attaching a handle of index i, or by attaching an i-handle, provided there is an embedding f: $\partial B^i \times B^{n-i} \to \partial W^n$ so that M^n is the adjunction space $W^n \cup_f (B^i \times B^{n-i})$. In this case we write $M^n = W^n \cup H^i$. The image of f is the attaching tube of H^i. The images of $B^i \times \{0\}$ and $\{0\} \times B^{n-i}$ in this quotient space are called the core and cocore, respectively, for H^i.

If $M^n = (W^n \cup H^i) \cup H^{i+1}$, we say H^i and H^{i+1} are complementary provided the core of H^{i+1} meets the cocore of H^i transversely in a single point. In this case M^n is (piecewise linearly or smoothly) homeomorphic to W^n.

The n-manifold M^n is said to have a k-dimensional spine, or k-spine, provided

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there exists a k-complex K and a map $g\colon \partial M^n \to K$ so that M^n is the mapping cyclinder of g.

For further reference see [2].

Proof of Theorem. We first observe that if M^4 does have a 2-spine, then M^4 is homotopy-equivalent to a 2-complex and hence $H_3(M^4) = 0$.

Conversely, let $M^4 = (B^3 \times I) \cup H_1^2 \cup \cdots \cup H_\beta^2 \cup \cdots \cup H_1^3 \cup \cdots \cup H_\gamma^3$, with f_k^i denoting the attaching map of the kth i-handle, and assume $H_3(M^4) = 0$. By general position, there is an isotopy of the f_k^i yielding:

(1) $[(\cup_{r=1}^{\beta} H_r^2) \cup (\cup_{s=1}^{\gamma} H_s^3)] \cap B^3 \times I \subset \operatorname{int}(B^3 \times \{1\})$,

(2) For each $k = 1, \ldots, \gamma$, there is a point $p_k \in \partial B^3$ such that

$$f_k^3(\{p_k\} \times B^1) \subset (B^3 \times \{1\}) - \left(\bigcup_{r=1}^{\beta} H_r^2\right).$$

Next, choose 2γ distinct points q_{kl}, $k = 1, \ldots, \gamma$, $l = -1, 1$, on the boundary of $B^3 \times \{1\}$. By duality, $H_3(M^4) = 0$ implies ∂M^4 is path-connected. Hence, for each pair k, l there exists an arc $\alpha_{kl} \subset \partial M^4$ beginning at q_{kl} and terminating at $f_k^3((p_k, l))$. We may isotope the α_{kl} so that:

(3) The α_{kl} are disjoint properly embedded arcs in

$$B^3 \times \{1\} - \left[\left(\bigcup_{r=1}^{\beta} H_r^2\right) \cup \left(\bigcup_{s=1}^{\gamma} \operatorname{int} f_s^3(\partial B^3 \times B^1)\right)\right],$$

(4) $\alpha_{kl} \cap (\cup_{s=1}^{\gamma} H_s^3) = f_k^3((p_k, l))$.

Let $\lambda_k = \alpha_{k,-1} \cup f_k^3(\{p_k\} \times B^1) \cup \alpha_{k,1}$. Then

(5) λ_k is an arc, properly embedded in $B^3 \times \{1\} - (\cup_{r=1}^{\beta} H_r^2)$,

(6) $\lambda_k \cap (\cup_{s=1}^{\gamma} H_s^3) = f_k^3(\{p_k\} \times B^1)$,

(7) the λ_k are disjoint.

If $N(\lambda_k)$ denotes a small regular neighborhood of λ_k in $B^3 \times \{1\}$, there is a homeomorphism $h_k\colon (B^2 \times B^1, \{0\} \times B^1) \to (N(\lambda_k), \lambda_k)$. Then $M^3 = B^3 \times \{1\} - \cup_{k=1}^{\gamma} h_k((\operatorname{int} B^2) \times B^1)$ is a compact 3-manifold with boundary. Therefore, M^3 has a 2-spine and hence a handle decomposition of 0-, 1- and 2-handles. Observe further that to regain $B^3 \times \{1\}$ from M^3 we attach γ 2-handles, namely the $N(\lambda_k)$. It follows from (5) that the attaching tubes of the H_i^2 lie in $M^3 \subset B^3 \times \{1\}$. Since crossing with the interval does not change the index of a handle, $W^4 = (M^3 \times I) \cup H_1^2 \cup \cdots \cup H_\beta^2$ has a 2-spine. Note that

$$M^4 = W^4 \cup \left[(N(\lambda_1) \times I) \cup H_1^3\right] \cup \cdots \cup \left[(N(\lambda_\gamma) \times I) \cup H_\gamma^3\right].$$

By (6), we obtain $N(\lambda_k) \times I$ is complementary to H_k^3 for $k = 1, \ldots, \gamma$ and hence $M^4 = W^4$.

References

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