A TWO POINT BOUNDARY VALUE PROBLEM WITH JUMPING NONLINEARITIES

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Abstract. We prove that a certain two point BVP with jumping nonlinearities has a solution. Our result generalizes that of [2]. We use variational methods which permit giving a minimax characterization of the solution. Our proof exposes the similarities between the variational behavior of this problem and that of other semilinear problems with noninvertible linear part (see [5]).

1. Introduction and notations. Here we study the two point BVP

\[
\begin{cases}
    u''(t) + g(u(t)) = p(t), & t \in [0, \pi], \\
    u(0) = u(\pi) = 0.
\end{cases}
\]

We assume that \( g: \mathbb{R} \to \mathbb{R} \) and \( p: [0, \pi] \to \mathbb{R} \) are continuous functions such that

(1.1) \( g(u) = u \) for \( u > 0 \).
(1.2) There exists \( a > 0 \) such that \( g(u)/u \to 1 + a \) as \( u \to -\infty \).
(1.3) \( \int_0^\pi p(t)\sin(t) \, dt < 0 \).

The purpose of this paper is to give a variational proof of

**Theorem A.** If (1.1), (1.2) and (1.3) are satisfied then (I) has a solution.

Theorem A is a generalization of a result due to L. Aguinaldo and K. Schmitt (see [2]). The main difference between our approach and that of [2] is that we use variational methods while the proof of [2] is based on degree theoretical arguments. As a byproduct of our technique for proving Theorem A we observe the functional \( J \), to be defined below, has a variational behavior similar to that of the functional corresponding to other semilinear problems with noninvertible linear part. We use a variant of a minimax principle proved first by P. Rabinowitz to obtain a variational proof of the theorem due to Ahmad, Lazer and Paul (see [3]).

Let \( H = H^1_0[0, \pi] \) (see [1, p. 44]) be the Sobolev space of square integrable functions defined on \( [0, \pi] \) vanishing on \( \{0, \pi\} \) with generalized first derivative in \( L^2[0, \pi] \). The inner product and norm in \( H \) are given by

\[
\langle u, v \rangle = \int_0^\pi u'(t)v'(t) \, dt \quad \text{and} \quad ||u||^2 = \langle u, u \rangle.
\]

According to Sobolev’s lemma (see [1, p. 95]) \( H \) can be imbedded in the space of continuous functions defined on \( [0, \pi] \). Thus, there exists a real number \( c > 0 \) such

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that

$$\max_{t \in [0, \pi]} |u(t)| < c\|u\| \quad \text{for all } u \in H. \quad (1.4)$$

We let $J: H \rightarrow \mathbb{R}$ be defined by

$$J(u) = \int_0^\pi \left( \frac{(u'(t))^2}{2} - G(u(t)) + p(t)u(t) \right) dt, \quad (1.5)$$

where $G(u) = \int_0^u g(s) \, ds$. It is easy to check that

$$\langle \nabla J(u), v \rangle \equiv \lim_{t \to 0} \frac{J(u + tv) - J(u)}{t} = \int_0^\pi u'v' - g(u)v + pv \quad \text{for all } u, v \in H. \quad (1.6)$$

By standard regularity theory it follows that if $\nabla J(u) = 0$ then $u$ is a solution of (I). Therefore, from now on we aim our work towards proving that $J$ has a critical point. In the rest of this paper the symbol $\int$ means integral from 0 to $\pi$ unless the integration limits are specified. For future reference, we remark that because of (1.1) and (1.2) there exists a real number $M_1$ such that

$$|G(u)| < ((1 + 2\alpha)u^2)/2 + M_1 \quad \text{for all } u \in \mathbb{R}. \quad (1.7)$$

2. Preliminary lemmas. If $\int p(t)\sin(t) \, dt = 0$ then it is easily verified that $\{u'' + u = p(t), t \in [0, \pi], u(0) = u(\pi) = 0\}$ has a positive solution $u_0$. Therefore $u_0$ is a solution of (I). Thus it is sufficient to restrict ourselves to the case

$$\int p(t)\sin(t) \, dt < 0. \quad (2.1)$$

**Lemma 1.** The functional $J$ satisfies $J(\lambda \sin(t)) \to -\infty$ as $|\lambda| \to \infty$.

**Proof.** That $J(\lambda \sin(t)) \to -\infty$ as $\lambda \to \infty$ follows immediately from (1.1) and (2.1). Because of (1.2) there exists a real number $M$ such that

$$G(u) > ((1 + \alpha/2)u^2)/2 + M \quad \text{for all } u < 0. \quad (2.2)$$

Therefore, for $\lambda < 0$ we have

$$J(\lambda \sin(t)) > (\lambda^2/2) \int \cos^2(t) \, dt - (1 + \alpha/2)\left( \int \sin^2(t) \right)(\lambda^2)/2$$

$$- M\pi + \lambda \int p(t)\sin(t) \, dt. \quad (2.3)$$

Since $\alpha > 0$, it is clear that (2.3) implies that $J(\lambda \sin(t)) \to -\infty$, as $\lambda \to -\infty$, and the lemma is proved.

We let $Y$ be the closed subspace of $H$ generated by $\{\sin(2t), \sin(3t), \ldots \}$. It is readily verified that $Y$ is the orthogonal complement of the subspace generated by $\sin(t)$. Observe that for $y \in Y$

$$4 \int y^2(t) \, dt < \int (y'(t))^2 \, dt. \quad (2.4)$$
Lemma 2. There exist real numbers $r_0 > 0$ and $\rho_0 > 0$ such that if $y \in Y$ and $\|y\| = r_0$ then
\[
J(\rho_0 \sin(t) + y(t)) > \sup_{\lambda \in \mathbb{R}} J(\lambda \sin(t)) + 1.
\]

Proof. Because of (1.4), given $\varepsilon > 0$ there exists $\rho > 0$ such that if $y \in Y$ and $\|y\| = 1$ then
\[
y(t) + \rho \sin(t) > 0 \quad \text{for all } t \in [\varepsilon, \pi - \varepsilon].
\]
Furthermore, for all $t \in [0, \pi]$, $y(t) + \rho \sin(t) > -c$. Consequently, for $b > 0$ and $y \in Y$ with $\|y\| = 1$,
\[
J(b(\rho \sin(t) + y(t))) > (b^2 \rho^2)/2 \int \cos^2(t) \, dt - (b^2 \rho^2)/2 \int \sin^2(t) \, dt
\]
\[
+ b^2/2 - b^2 \int y^2(t) \, dt + b \rho \int p(t) \sin(t) \, dt
\]
\[
+ b \int p(t)y(t) \, dt - \int_{\pi - \varepsilon}^{\varepsilon} G(b(\rho \sin(t) + y(t))) \, dt
\]
\[
- \int_{\pi - \varepsilon}^{\varepsilon} G(b(\rho \sin(t) + y(t))) \, dt.
\]
Combining this with (2.2) and (2.4) we have
\[
J(b(\rho \sin(t) + y(t))) > b^2(3/8) - b \rho \int p(t) \sin(t) \, dt
\]
\[
- 2(1 + (\alpha/2))b^2c^2 - 2eM.
\]
Thus, choosing $\varepsilon$ small enough and $b$ sufficiently large we see that $r_0 = b$ and $\rho_0 = \rho b$ satisfy the conditions of the lemma. Hence, Lemma 2 is proved.

From now on $\rho_0$ and $r_0$ denote two fixed real numbers satisfying Lemma 2. Because of (1.1) and (1.2), $y$ is bounded on bounded sets. Therefore, there exists a real number $c_1$ such that if $y \in Y$ and $\|y\| = r_0$ then
\[
J(\rho_0 \sin(t) + y(t)) > c_1. \quad (2.5)
\]
We let $\lambda_0 > \rho_0$ be such that
\[
\max\{J(\lambda_0 \sin(t)), J(-\lambda_0 \sin(t))\} < c_1 - 1. \quad (2.6)
\]
We denote by $\Sigma$ the family of all continuous functions $\sigma: [0, 1] \to S \equiv H - (\rho_0 \sin(t) + y(t); y \in Y$ and $\|y\| = r_0$) such that
(a) $\sigma(0) = -\lambda_0 \sin(t)$, $\sigma(1) = \lambda_0 \sin(t)$, and
(b) $\sigma$ is homotopic on $S$ to a map $\sigma_0$ through a homotopy which keeps end points fixed, where $\sigma_0$ is defined by $\sigma_0(s) = 2s\lambda_0 \sin(t) - \lambda_0 \sin(t)$.

An elementary topological argument shows that if $\sigma \in \Sigma$ then there exists $s \in [0, 1]$ and $y \in Y$ with $\sigma(s) = \rho_0 \sin(t) + y(t)$ and $\|y\| < r_0$.

Theorem 3. Let $J$, $r_0$, $\rho_0$, and $\Sigma$ be as before. If every sequence $\{x_n\} \subset H$ such that $\nabla J(x_n) \to 0$ and $\{J(x_n)\}$ is bounded has a convergent subsequence, then $J$ has a critical point $u_0$. Moreover
\[
J(u_0) = \inf_{\sigma \in \Sigma} \left( \max_{s \in [0,1]} J(\sigma(s)) \right).
\]
Since Theorem 3 is a slight variant of Theorem 1.2 of [5] we do not give a proof of it here.

3. Proof of Theorem A. Let \( \{x_n\} \subseteq H \) be a sequence such that \( \nabla J(x_n) \to 0 \) and \( \{J(x_n)\} \) is bounded. According to Theorem 3 and the remark following (1.6) we only need show that \( \{x_n\} \) has a convergent subsequence.

By (1.6), \( \nabla J(x) = x + g_1(x) \), where \( g_1: H \to H \) is continuous. Moreover, since the inclusion of \( H \) into \( L_2[0, \pi] \) is compact (see [1, Theorem 6.2]), \( g_1 \) is compact. In addition, \( g_1 \) maps weakly convergent sequences into convergent sequences.

Suppose \( \{x_n\} \) does not have a convergent subsequence. First we observe that \( \{x_n\} \) does not have a weakly convergent subsequence \( \{x^*_n\} \). For if it does, then \( \{g_1(x_n)\} \) is a convergent sequence; since \( x_n + g_1(x_n) \to 0 \) we have that \( \{x_n\} \) converges, a contradiction. Hence we can assume that \( \|x_n\| \) tends to \( +\infty \).

Let \( \{x_n/\|x_n\|\} \) be a subsequence of \( \{x_n/\|x_n\|\} \) converging weakly to a point \( x_0 \) in \( H \). For each \( v \in H \) we have
\[
\langle \nabla J(x_n), v \rangle /\|x_n\| = \left( \int x'_n v' - g(x_n) v + pv \right) /\|x_n\| \to 0.
\]
Therefore
\[
\int \left( x'_n v' - \left( g(x_n) /\|x_n\| \right) v \right) \to 0 \quad \text{as } j \to \infty. \tag{2.7}
\]
Because of (1.1) and (1.2) we have \( g(x) = f_1(x) + f_2(x) \) where \( f_1 \) is defined by \( f_1(x) = x \) for \( x > 0 \) and \( f_1(x) = (1 + \alpha)x \) for \( x < 0 \), and \( f_2 \) satisfies
\[
(f_2(x))/x \to 0 \quad \text{as } |x| \to +\infty. \tag{2.8}
\]
Thus, we have
\[
\int \left( g(x_n) /\|x_n\| \right) v = \int f_1(x_n/\|x_n\|) v + (f_2(x_n) v) /\|x_n\|.
\]
Because of (2.8) and (2.9) we obtain
\[
\int x'_n v' - f_1(x_n) v = 0 \quad \text{for all } v \in H.
\]
Therefore \( x_0 \) satisfies \( x'_0 + f_1(x_0) = 0 \), \( x(0) = x(\pi) = 0 \), which implies that \( x_0 = \xi \sin(t) \) for some \( \xi > 0 \). In case \( \xi = 0 \), by (1.7) we have that given \( \epsilon > 0 \) there exists \( j_0 \) such that
\[
\int G(x_n(t)) dt < \epsilon(1 + 2\alpha)\|x_n\|^2/2 + M_1\pi \quad \text{for all } j > j_0. \tag{2.9}
\]
From this inequality we have
\[
J(x_n) > \|x_n\|^2/2 - \epsilon(1 + 2\alpha)\|x_n\|^2/2 - M_1\|x_n\| - \|x_n\| \int (px_n) /\|x_n\|. \tag{2.10}
\]
When \( \epsilon(1 + 2\alpha) < 1 \) our assumption that \( \|x_n\| \to \infty \) contradicts the boundedness of \( \{\|J(x_n)\|\} \).
It remains to consider the case $\xi > 0$. Note that $(g(x_n)/\|x_n\|)$ converges in $L^2[0, \pi]$ to $f_1(x_0)$ and $g_1(x_n)/\|x_n\|$ converges to $g_1(x_0)$. We are assuming $\nabla J(x_n) = x_n + g_1(x_n) \to 0$, so $(x_n/\|x_n\|)$ converges to $x_0$. Since $x_0$ is positive on $(0, \pi)$ with $x'(0) > 0$ and $x'(\pi) < 0$, we have that there exists $j_1$ and a real number $c$ such that $x_n(t) > c$ for all $t \in [0, \pi]$ and all $j > j_1$. Thus, as $j \to \infty$,
\[
\frac{(2J(x_n))/\|x_n\| = \int \left( (x_n')^2 + 2px_n \right)/\|x_n\|} = 0,
\]
and
\[
\langle \nabla J(x_n), x_n/\|x_n\| \rangle = \int \left( (x_n')^2 + px_n \right)/\|x_n\| \nabla J(x_n) + \int (g(x_n)x_n)/\|x_n\| \to 0.
\]
By the hypotheses on $(x_n)$ we find $(\int p(x_n)/\|x_n\|) \to 0$. Then by (2.1), $\xi$ cannot be positive. This final contradiction implies that $(x_n)$ has a convergent subsequence and Theorem A is proved.

Remark. With obvious modifications of the method above one can prove results analogous to Theorem A for other type of boundary conditions. For example, it can be shown that

$$(II) \quad \begin{cases} u'' + g(u(t)) = p(t), & t \in [0, \pi], \\ u(0) = u(\pi) = 0 \end{cases}$$

has a solution if

$(1') g(u) = 0$ for $u > 0$.

$(11')$ for some $\alpha > 0, g(u)/u \to \alpha$ as $u \to -\infty$.

$(111') \int p(t) \, dt < 0$.

References


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