THE COMPACT HANKEL OPERATORS FORM
AN M-IDEAL IN THE SPACE OF HANKEL OPERATORS

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Abstract. The theorem in the title is proved and used to give a new proof that elements of \( L^\infty \) have best approximations in \( H^\infty + C \).

1. Introduction. Let \( L^\infty(|z| = 1, d\theta) \) denote the usual Lebesgue space of functions on the unit circle. Let \( H^\infty \) denote the subalgebra consisting of boundary values of bounded analytic functions on \( |z| < 1 \), and \( H^\infty + C \) the linear span of \( H^\infty \) and the continuous functions on \( |z| = 1 \). In [3] Axler, et al. prove the following theorem.

**Theorem 1.1.** Let \( f \in L^\infty \). Then there exists \( h \in H^\infty + C \) such that \( \|f - h\|_\infty = \text{dist}(f, H^\infty + C) \).

Here \( \| \cdot \|_\infty \) denotes the essential supremum norm and the distance is measured in this norm.

In this paper we prove the result in the title, which yields 1.1 as a corollary.

2. Preliminaries and main theorem.

**Definition.** A subspace \( Y \) of a Banach space \( X \) is called an \( L \)-ideal if there is projection \( E: X \to Y \) such that

\[
\|y\| = \|Ey\| + \|y - Ey\|, \quad y \in Y.
\]

Such an \( E \) is called an \( L \)-projection. A subspace \( K \) of a Banach space \( X \) is called an \( M \)-ideal if the annihilator \( K^\perp \) is an \( L \)-ideal of \( X^* \).

These concepts may be found in Alfsen and Effros [2]. The property of \( M \)-ideals we are concerned with is the following.

**Theorem 2.1.** If \( K \) is an \( M \)-ideal of \( X \) and if \( x \in X \), then there exists \( m \in K \) such that \( \|x - m\| = \text{dist}(x, K) \).

**Proof.** This is an immediate corollary of Corollary 5.6 of [2]. Although [2] concerns only real Banach spaces and real linear operators one need only consider the underlying real structure to obtain the same result for complex Banach spaces.

We let \( H^2 \) denote the usual Hardy space of functions in \( L^2(|z| = 1, d\theta) \) with vanishing negative Fourier coefficients, and \( (H^2)^\perp \) the orthogonal complement of \( H^2 \) in \( L^2 \). For \( f \in L^\infty \) the Hankel operator \( H_f: H^2 \to (H^2)^\perp \) is defined by

\[
H_fg = P(fg), \quad g \in H^2,
\]

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where $P$ is the orthogonal projection onto $(H^2)^\perp$. We need only the following properties of Hankel operators.

**Theorem 2.2.** The norm of $H_f$ is given by $\|H_f\| = \operatorname{dist}(f, H^\infty)$. The essential norm of $H_f$ (distance to the compacts) is given by $\|H_f\|_e = \operatorname{dist}(f, H^\infty + C)$. In particular, $H_f$ is compact if and only if $f \in H^\infty + C$.

The first part is due to Nehari [7], the second follows from Theorem 0.1 of [1] (see also [3, p. 608]).

The following is our major result.

**Theorem 2.3.** The space of compact Hankel operators forms an M-ideal in the space of all Hankel operators.

Using Theorem 2.2 this may be restated.

**Theorem 2.4.** $(H^\infty + C)/H^\infty$ is an M-ideal in the Banach space $L^\infty/H^\infty$.

**Proof.** The map which takes $f + H^\infty$ to $H_f$ is an isometry of $L^\infty/H^\infty$ onto the space of Hankel operators that takes $(H^\infty + C)/H^\infty$ onto the compact Hankel operators.

We take time out here to show how Theorem 1.1 follows from Theorem 2.1 and Theorem 2.4. These imply that for any $f \in L^\infty$ there exists $g \in H^\infty + C$ such that $\operatorname{dist}(f, H^\infty + C) = \operatorname{dist}(f - g, H^\infty)$. Since $H^\infty$ is weak* closed in $L^\infty$, a compactness argument yields $h_0 \in H^\infty$ such that $\operatorname{dist}(f - g, H^\infty) = \|f - g - h_0\|_\infty$. This gives us Theorem 1.1 with $h = g + h_0$.

3. **Proof of the theorem.** We assume known a number of standard results in the theory of function algebras (particularly logmodular algebras). Excellent sources are [4], [5], and [6].

To prove the main theorem (in the form Theorem 2.4) we make a number of canonical identifications.

We identify $L^\infty$ with $C(M)$ where $M$ is the maximal ideal space (Gelfand space) of $L^\infty$. When it is necessary to make a distinction we let $\hat{f} \in C(M)$ denote the function identified with $f \in L^\infty$ via the Gelfand transformation.

The dual space $(L^\infty/H^\infty)^*$ is identified with the space

$$(H^\infty)^\perp = \left\{ \mu \in C(M)^*: \int f \, d\mu = 0 \text{ for all } f \in H^\infty \right\}$$

where $H^\infty$ is viewed as a subalgebra of $C(M)$. As such it is logmodular (see [5]).

We identify $((H^\infty + C)/H^\infty)^\perp$ with $(H^\infty + C)^\perp = \left\{ \mu \in C(M)^*: \int f \, d\mu = 0 \text{ for all } f \in H^\infty + C \right\}$. We point out that $H^\infty + C$ is a closed subalgebra of $L^\infty$. (See [8, p. 191].)

Let $m$ denote the lifting of Lebesgue measure from the unit circle to $M$. That is,

$$\int_M \hat{f} \, dm = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \, dt, \quad \text{for } f \in L^\infty. \quad (3.1)$$

This induces an isomorphism of $L^1(|z| = 1, d\theta)$ with $L^1(M, dm)$ in which (3.1) holds for $f \in L^1$. 

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To prove Theorem 2.4 we have to produce an L-projection of \((H^{\infty})^\perp\) onto \((H^{\infty} + C)^\perp\). It is constructed as follows. A measure \(\mu \in (H^{\infty})^\perp\) can be written \(\mu = \mu_a + \mu_s\) where \(\mu_a\) is absolutely continuous with respect to \(m\) and \(\mu_s\) is singular to \(m\). Define \(E\mu = \mu_a\). It must be verified that \(E\) is the required L-projection.

It is clear that \(\|\mu\| = \|E\mu\| + \|\mu - E\mu\|\) so we only have to show \(E(H^{\infty})^\perp = (H^{\infty} + C)^\perp\). The Abstract F. and M. Riesz Theorem, as given in [4, p. 44, Corollary 7.9], takes the following form when applied to \(H^{\infty}\).

**Theorem 3.2.** Let \(H_0^{\infty} = \{f \in H^{\infty}: \int f\, dm = 0\}\) and \((H_0^{\infty})^\perp = \{\mu \in C(M)^\ast: \int f\, d\mu = 0\ for all f \in H_0^{\infty}\}\). Let \(\mu \in (H_0^{\infty})^\perp\) be written \(\mu = \mu_a + \mu_s\) where \(\mu_a \ll m\) and \(\mu_s \perp m\); then \(\mu_a \in (H_0^{\infty})^\perp\) and \(\mu_s \in (H^{\infty})^\perp\).

Now, if \(\mu \in (H^{\infty})^\perp\) then \(\tilde{\mu} \in (H_0^{\infty})^\perp\), where \(\tilde{\mu}\) is the (Gelfand transform of) the function on the unit circle which takes \(e^{i\theta}\) to \(e^{-i\theta}\). If \(\mu\) is singular with respect to \(m\) so is \(\tilde{\mu}\) and Theorem 3.2 implies \(\tilde{\mu} \in (H^{\infty})^\perp\). Repetition of this argument gives \(\tilde{\mu} \in (H^{\infty})^\perp\), that is

\[
\int \tilde{\mu} f\, dm = 0 \quad \text{for all } f \in H^{\infty}, \ n > 0.
\]

Since \(\bigcup_{n=0}^{\infty} \tilde{\mu}^n H^{\infty}\) is dense in \(H^{\infty} + C\) we see that \(\mu \in (H^{\infty} + C)^\perp\) provided \(\mu \in (H^{\infty})^\perp\) and \(\mu \perp m\). If \(\mu\) is an arbitrary element of \((H^{\infty})^\perp\) then Theorem 3.2 implies \(E\mu \in (H^{\infty})^\perp\). Since \(E\mu \perp m\) we get \(E\mu \in (H^{\infty} + C)^\perp\).

To show that \(E\) is onto we show that every \(\mu \in (H^{\infty} + C)^\perp\) is singular with respect to \(m\). Suppose \(\mu = \mu_a + \mu_s\) as before. Then \(\mu_s \in (H^{\infty} + C)^\perp\) again, and so \(\mu_s \in (H^{\infty} + C)^\perp\). Writing \(\mu_s = \hat{h}\, dm\) with \(h \in L^1(d\theta)\) this implies \(\int gh\, d\theta = 0\) for all continuous \(g\). Thus \(h = 0\) and \(\mu_s = 0\), i.e., \(\mu = E\mu\) and \(E\) is onto.

**4. Remarks.** It would be interesting to know if the analogues of Theorem 1.1 and Theorem 2.4 are true with \(H^{\infty} + C\) replaced by other subalgebras of \(L^\infty\) containing \(H^{\infty}\).

The best approximation guaranteed by Theorem 1.1 is not unique (in fact is never unique) if \(f \not\in H^{\infty} + C\). This was obtained in [3] and can also be deduced from Theorem 2.4 using results on \(M\)-ideals by Holmes, Scranton, and Ward [7].

**References**


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