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THE COMPACT HANKEL OPERATORS FORM AN M-IDEAL IN THE SPACE OF HANKEL OPERATORS

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ABSTRACT. The theorem in the title is proved and used to give a new proof that elements of L^∞ have best approximations in $H^\infty + C$.

1. Introduction. Let $L^\infty(|z| = 1, d\theta)$ denote the usual Lebesgue space of functions on the unit circle. Let H^∞ denote the subalgebra consisting of boundary values of bounded analytic functions on $|z| < 1$, and $H^\infty + C$ the linear span of H^∞ and the continuous functions on $|z| = 1$. In [3] Axler, et al. prove the following theorem.

THEOREM 1.1. *Let $f \in L^\infty$. Then there exists $h \in H^\infty + C$ such that $\|f - h\|_\infty = \operatorname{dist}(f, H^\infty + C)$.*

Here $\|\cdot\|_\infty$ denotes the essential supremum norm and the distance is measured in this norm.

In this paper we prove the result in the title, which yields 1.1 as a corollary.

2. Preliminaries and main theorem.

DEFINITION. A subspace J of a Banach space Y is called an *L-ideal* if there is projection E: $Y \to J$ such that

$$\|y\| = \|Ey\| + \|y - Ey\|, \qquad y \in Y.$$

Such an E is called an *L-projection*. A subspace K of a Banach space X is called an *M-ideal* if the annihilator $K^\perp$ is an *L-ideal* of X^*.

These concepts may be found in Alfsen and Effros [2]. The property of M-ideals we are concerned with is the following.

THEOREM 2.1. *If K is an M-ideal of X and if $x \in X$, then there exists $m \in K$ such that $\|x - m\| = \operatorname{dist}(x, K)$.*

PROOF. This is an immediate corollary of Corollary 5.6 of [2]. Although [2] concerns only real Banach spaces and real linear operators one need only consider the underlying real structure to obtain the same result for complex Banach spaces.

We let H^2 denote the usual Hardy space of functions in $L^2(|z| = 1, d\theta)$ with vanishing negative Fourier coefficients, and $(H^2)^\perp$ the orthogonal complement of H^2 in L^2. For $f \in L^\infty$ the Hankel operator H_f: $H^2 \to (H^2)^\perp$ is defined by

$$H_f g = P(fg), \qquad g \in H^2,$$

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where P is the orthogonal projection onto $(H^2)^\perp$. We need only the following properties of Hankel operators.

THEOREM 2.2. *The norm of H_f is given by $\|H_f\| = \operatorname{dist}(f, H^\infty)$. The essential norm of H_f (distance to the compacts) is given by $\|H_f\|_e = \operatorname{dist}(f, H^\infty + C)$. In particular, H_f is compact if and only if $f \in H^\infty + C$.*

The first part is due to Nehari [7], the second follows from Theorem 0.1 of [1] (see also [3, p. 608]).

The following is our major result.

THEOREM 2.3. *The space of compact Hankel operators forms an M-ideal in the space of all Hankel operators.*

Using Theorem 2.2 this may be restated.

THEOREM 2.4. *$(H^\infty + C)/H^\infty$ is an M-ideal in the Banach space L^∞/H^∞.*

PROOF. The map which takes $f + H^\infty$ to H_f is an isometry of L^∞/H^∞ onto the space of Hankel operators that takes $(H^\infty + C)/H^\infty$ onto the compact Hankel operators.

We take time out here to show how Theorem 1.1 follows from Theorem 2.1 and Theorem 2.4. These imply that for any $f \in L^\infty$ there exists $g \in H^\infty + C$ such that $\operatorname{dist}(f, H^\infty + C) = \operatorname{dist}(f - g, H^\infty)$. Since H^∞ is weak* closed in L^∞, a compactness argument yields $h_0 \in H^\infty$ such that $\operatorname{dist}(f - g, H^\infty) = \|f - g - h_0\|_\infty$. This gives us Theorem 1.1 with $h = g + h_0$.

3. Proof of the theorem. We assume known a number of standard results in the theory of function algebras (particularly logmodular algebras). Excellent sources are [4], [5], and [6].

To prove the main theorem (in the form Theorem 2.4) we make a number of canonical identifications.

We identify L^∞ with $C(M)$ where M is the maximal ideal space (Gelfand space) of L^∞. When it is necessary to make a distinction we let $\hat{f} \in C(M)$ denote the function identified with $f \in L^\infty$ via the Gelfand transformation.

The dual space $(L^\infty/H^\infty)^*$ is identified with the space

$$(H^\infty)^\perp = \left\{ \mu \in C(M)^*\colon \int f\, d\mu = 0 \text{ for all } f \in H^\infty \right\}$$

where H^∞ is viewed as a subalgebra of $C(M)$. As such it is logmodular (see [5]).

We identify $((H^\infty + C)/H^\infty)^\perp$ with $(H^\infty + C)^\perp = \{\mu \in C(M)^*\colon \int f\, d\mu = 0$ for all $f \in H^\infty + C\}$. We point out that $H^\infty + C$ is a closed subalgebra of L^∞. (See [8, p. 191].)

Let m denote the lifting of Lebesgue measure from the unit circle to M. That is,

$$\int_M \hat{f}\, dm = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta, \quad \text{for } f \in L^\infty. \tag{3.1}$$

This induces an isomorphism of $L^1(|z| = 1, d\theta)$ with $L^1(M, dm)$ in which (3.1) holds for $f \in L^1$.

To prove Theorem 2.4 we have to produce an L-projection of $(H^\infty)^\perp$ onto $(H^\infty + C)^\perp$. It is constructed as follows. A measure $\mu \in (H^\infty)^\perp$ can be written $\mu = \mu_a + \mu_s$ where μ_a is absolutely continuous with respect to m and μ_s is singular to m. Define $E\mu = \mu_s$. It must be verified that E is the required L-projection.

It is clear that $\|\mu\| = \|E\mu\| + \|\mu - E\mu\|$ so we only have to show $E(H^\infty)^\perp = (H^\infty + C)^\perp$. The Abstract F. and M. Riesz Theorem, as given in [4, p. 44, Corollary 7.9], takes the following form when applied to H^∞.

THEOREM 3.2. *Let* $H_0^\infty = \{f \in H^\infty: \int f\, dm = 0\}$ *and* $(H_0^\infty)^\perp = \{\mu \in C(M)^*: \int f\, d\mu = 0$ *for all* $f \in H_0^\infty\}$. *Let* $\mu \in (H_0^\infty)^\perp$ *be written* $\mu = \mu_a + \mu_s$ *where* $\mu_a \ll m$ *and* $\mu_s \perp m$; *then* $\mu_a \in (H_0^\infty)^\perp$ *and* $\mu_s \in (H^\infty)^\perp$.

Now, if $\mu \in (H^\infty)^\perp$ then $\bar{z}\mu \in (H_0^\infty)^\perp$, where $\bar{z}$ is the (Gelfand transform of) the function on the unit circle which takes $e^{i\theta}$ to $e^{-i\theta}$. If μ is singular with respect to m so is $\bar{z}\mu$ and Theorem 3.2 implies $\bar{z}\mu \in (H^\infty)^\perp$. Repetition of this argument gives $\bar{z}^n\mu \in (H^\infty)^\perp$, that is

$$\int \bar{z}^n f\, d\mu = 0 \quad \text{for all } f \in H^\infty, n > 0.$$

Since $\bigcup_{n=0}^\infty \bar{z}^n H^\infty$ is dense in $H^\infty + C$ we see that $\mu \in (H^\infty + C)^\perp$ provided $\mu \in (H^\infty)^\perp$ and $\mu \perp m$. If μ is an arbitrary element of $(H^\infty)^\perp$ then Theorem 3.2 implies $E\mu \in (H^\infty)^\perp$. Since $E\mu \perp m$ we get $E\mu \in (H^\infty + C)^\perp$.

To show that E is onto we show that every $\mu \in (H^\infty + C)^\perp$ is singular with respect to m. Suppose $\mu = \mu_a + \mu_s$ as before. Then $\mu_s \in (H^\infty + C)^\perp$ again, and so $\mu_a \in (H^\infty + C)^\perp$. Writing $\mu_a = \hat{h}dm$ with $h \in L^1(d\theta)$ this implies $\int gh\, d\theta = 0$ for all continuous g. Thus $h = 0$ and $\mu_a = 0$, i.e., $\mu = E\mu$ and E is onto.

4. Remarks. It would be interesting to know if the analogues of Theorem 1.1 and Theorem 2.4 are true with $H^\infty + C$ replaced by other subalgebras of L^∞ containing H^∞.

The best approximation guaranteed by Theorem 1.1 is not unique (in fact is never unique) if $f \notin H^\infty + C$. This was obtained in [3] and can also be deduced from Theorem 2.4 using results on M-ideals by Holmes, Scranton, and Ward [7].

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