ZEROS OF SUCCESSIVE DERIVATIVES AND ITERATED OPERATORS ON ANALYTIC FUNCTIONS

J. K. SHAW AND C. L. PRATHER

Abstract. For a function f analytic in the closed disc \(|z| < 1\), we study the behavior of zeros of the successive iterates \((\theta^n f)(z)\), \(n = 0, 1, 2, \ldots\), where \(\theta = (z + \alpha)^{p+1} d/dz\). We find that such behavior closely parallels that for the ordinary derivative operator. Using change-of-variable methods, we obtain information on zeros of derivatives of functions analytic in half-planes.

1. Introduction. Let f be analytic in the unit disc \(|z| < 1\). A well-known principle in function theory is that f cannot have too many derivatives vanishing too near \(z = 0\), unless f is a polynomial. The study of this phenomenon is the theory which has been associated with the names of Gončarov and Whittaker [2]–[5]. Its principal feature is the existence of zero-free neighborhoods of \(z = 0\). There is an absolute constant \(G\), known as the Gončarov constant, such that if \(f\) is analytic in \(|z| < 1\), is not a polynomial, and \(\epsilon > 0\), then there is an infinite sequence of derivatives \(f^{(n)}(z)\) which do not vanish in the discs \(|z| < (G - \epsilon)/(n_k + 1)\). The exact value of \(G\) is unknown, but it is known to lie between .7259 and .7378.

In the present paper, we consider the analogous problem for the case of the differential operator \(\theta = (z + \alpha)^{p+1} d/dz\), where \(p > 0\) and where \(\alpha\) denotes a complex number. Taking \(|\alpha| < 1\) and \(f(z)\) analytic in a neighborhood of the closed unit disc \(|z| < 1\), we study the zero-free regions of the iterates \((\theta^n f)(z)\), \(n = 0, 1, 2, \ldots\). The neighborhoods of \(z = -\alpha\) are the most interesting, for in this case all but a finite number (not just an infinite number) of the iterates \((\theta^n f)(z)\) are nonzero in punctured discs which shrink with increasing \(n\) to the point \(z = -\alpha\).

The results we obtain for differentiation do not arise from taking \(p = -1\) in the definition of \(\theta\). Instead, we use other values of \(p\) and employ change-of-variable methods to get information about zeros of derivatives of functions analytic in regions of the plane other than discs. Such problems have been studied by Widder [7, Theorem 31 and corollary, pp. 166–167] for functions analytic at \(\infty\), and for functions analytic in half-planes and representable as Laplace Transforms, and a simpler proof of a result implied by Widder has recently been given by Boas [1]. This result can be stated as follows.

**Theorem A** ([1], [7]). Let \(F(w) = \sum_{n=-\infty}^{\infty} \beta_n w^{-n}\) be analytic at \(\infty\), with \(F\) nonconstant. Then there is a constant \(c > 0\) such that for all \(n\) sufficiently large, \(F^{(n)}(w)\) has no finite zero outside the circle \(|w| = nc\).
As \( F \) is analytic in \(|z| > R\) in Theorem A, the constant \( c \) depends on \( F \) and \( R \).

Our results yield this theorem as a special case. We also obtain Widder’s description of the radial distribution of zeros [7, Theorem 35], and we get additional theorems on periodic functions, where Widder’s method does not apply. These applications are given in §3.

2. The \( \theta \)-operator. We shall obtain an integral representation for the \( n \)th iterate \((\theta^n f)(z)\). There are various ways of getting to the end result; for example, one could extend the method used by Hille [6, Vol. 2, p. 51] for the operator \( z \, d/dz \). We will use an alternate approach.

We do this by expressing \( \theta^n[z^k] \) as a function of \((z + \alpha)^{1+p}\) having polynomial coefficients, as given in (2.2), and then express the polynomials as integral transforms. Then the integral representation for \( \theta^n f \) is obtained by the usual power series method.

For each nonnegative integer \( m \), we have \( \theta[(z + \alpha)^m] = m(z + \alpha)^{m+p} \), and more generally,

\[
\theta^n[(z + \alpha)^m] = (m)(m + p)(m + 2p) \cdots (m + (n - 1)p)(z + \alpha)^{m+p} = C_{mn}^{(p)}(z + \alpha)^{m+p}, \quad m > 0, n > 1,
\]

(2.1)

where \( C_{mn}^{(p)} = (m)(m + p)(m + 2p) \cdots (m + (n - 1)p) \). If we apply (2.1) to the binomial expansion

\[
z^k = \sum_{m=0}^{k} (-1)^m \binom{k}{m}(z + \alpha)^{k-m}\alpha^m,
\]

and then put \( z + \alpha = -\xi \), we get

\[
\theta^n(z^k) = \sum_{m=0}^{k-1} (-1)^m \binom{k}{m} C_{k-m,n}^{(p)}(z + \alpha)^{k-m+p}\alpha^m
\]

\[
= (z + \alpha)^{1+p} \sum_{m=0}^{k-1} (-1)^m \binom{k}{m} \alpha^m C_{k-m,n}^{(p)} \xi^{k-m-1}
\]

\[
= (z + \alpha)^{1+p} (-1)^{k-1} \left\{ \sum_{m=0}^{k-1} \binom{k}{m} \alpha^m C_{k-m,n}^{(p)} \xi^{k-m-1} \right\}
\]

\[
= (z + \alpha)^{1+p} (-1)^{k-1} P_{k-1}^{(n)}(\xi), \quad n > 1, k > 1,
\]

(2.2)

with \( P_{k-1}^{(n)}(\xi) \) defined in the indicated way. Of course, \( P_{k-1}^{(n)}(\xi) \) also depends on \( \alpha \) and \( p \), but we suppress this dependence to simplify notation. From the definition of \( C_{mn}^{(p)} \) we see that

\[
0 < C_{mn}^{(p)} < m^n[1 + (n - 1)p]^n,
\]

(2.3)

and so

\[
|P_{k-1}^{(n)}(\xi)| < [1 + (n - 1)p]^n \sum_{m=0}^{k} \binom{k}{m}|\alpha|^m(k - m)^n|\xi|^{k-m-1}
\]

\[
< |\xi|^{-1}[1 + (n - 1)p]^n k^n(|\alpha| + |\xi|)k, \quad n > 1, k > 1.
\]
For each fixed $\xi$, then, the power series

$$\sum_{k=1}^{\infty} P_{k-1}^{(n)}(\xi) t^{k-1}$$

(which also depends on $\alpha$ and $p$) converges at least in the disc $|t| < (|\alpha| + |\xi|)^{-1}$. Now substitute the defining expression for $P_{k-1}^{(n)}(\xi)$ into (2.4) and formally interchange the order of summation. This leads to

$$\sum_{k=1}^{\infty} P_{k-1}^{(n)}(\xi) t^{k-1} = \sum_{k=1}^{\infty} \left\{ \sum_{r=0}^{k-1} \binom{k}{r} \alpha^r C_{k-r, \alpha}^{(p)} t^{k-r-1} \right\} t^{k-1}$$

$$= \sum_{k=1}^{\infty} t^{k-1} \sum_{m=1}^{k} \binom{k}{k-m} \alpha^{k-m} C_{m}^{(p)} t^{m-1}$$

$$= \sum_{m=1}^{\infty} C_{m}^{(p)} t^{m-1} \sum_{k=m}^{\infty} \binom{k}{k-m} \alpha^{k-m} t^{k-m}$$

$$= \sum_{m=1}^{\infty} C_{m}^{(p)}(\xi t)^{m-1} \sum_{r=0}^{\infty} \binom{m+r}{m} (\alpha t)^r$$

$$= \sum_{m=1}^{\infty} C_{m}^{(p)}(\xi t)^{m-1} (1 - \alpha t)^{-(m+1)}$$

$$= (1 - \alpha t)^{-2} \sum_{m=1}^{\infty} C_{m}^{(p)} \left[ \frac{\xi t}{(1 - \alpha t)^{m+1}} \right].$$

(2.5)

In view of (2.3) it follows that the interchange in order of summation will be justified when $|\alpha t| + |\xi| < 1$. Note that $|\alpha t| + |\xi| < 1$ implies $|\xi| < 1 - |\alpha t|$, which implies

$$\left| \frac{\xi t}{1 - \alpha t} \right| < \frac{1 - |\alpha t|}{|1 - \alpha t|} < 1.$$ 

In particular, (2.4) and (2.5) are both valid, and we have

$$G_n(\xi, t) = \sum_{k=1}^{\infty} P_{k-1}^{(n)}(\xi) t^{k-1} = (1 - \alpha t)^{-2} \sum_{m=1}^{\infty} C_{m}^{(p)} \left[ \frac{\xi t}{1 - \alpha t} \right]^{m-1},$$

for $|t| < 1/(|\alpha| + |\xi|)$. (2.6)

Let $\xi$ be fixed and let the real number $r$ satisfy $0 < r < (|\alpha| + |\xi|)^{-1}$. Then the Cauchy Integral Formula applied to (2.6) gives

$$P_{k-1}^{(n)}(\xi) = \frac{1}{2\pi i} \int_{|t| = r} \frac{G_n(\xi, t)}{t^k} \, dt, \quad n > 1, k > 1.$$ 

(2.7)

**Theorem 2.1.** Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be analytic in the disc $|z| < R$, where $R > 1$, let $r$ satisfy $1 < r^{-1} < R$, and let $\xi$ and $\alpha$ satisfy $|\xi| + |\alpha| < r^{-1}$. If $z + \alpha = -\xi$, then

$$\left( \frac{\theta^n}{2\pi i} \right) \int_{|t| = r} \frac{G_n(\xi, t)}{t^n} \, dt, \quad n = 1, 2, 3, \ldots$$

(2.8)
Proof. Using (2.2) and (2.7), apply \( \theta^n \) termwise to the power series for \( f(z) \) to obtain

\[
(\theta^n f)(z) = \sum_{k=1}^{\infty} a_k \theta^n (z^k) = (z + \alpha)^{1 + np} \sum_{k=1}^{\infty} (-1)^{k-1} a_k P_{k-1}^{(n)}(\zeta)
\]

\[
= (z + \alpha)^{1 + np} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a_k}{2\pi i} \int_{|t|=r} \frac{G_n(\zeta, t)}{t^k} \, dt.
\]

Since \( |t^{-1}| = r^{-1} < R \) in the range of integration, uniform convergence gives

\[
(\theta^n f)(z) = \frac{(z + \alpha)^{1 + np}}{2\pi i} \int_{|t|=r} \left\{ \sum_{k=1}^{\infty} \frac{(-1)^{k-1} a_k}{t^k} \right\} G_n(\zeta, t) \, dt
\]

\[
= \frac{(z + \alpha)^{1 + np}}{2\pi i} \int_{|t|=r} \left\{ f(0) - f\left(\frac{1}{t}\right) \right\} G_n(\zeta, t) \, dt.
\]

Since \( G_n(\zeta, t) \) is analytic for \( |t| < (|\alpha| + |\zeta|)^{-1} \), the term involving \( f(0) \) drops out, giving (2.8).

The representation (2.8) extends the analogous formula of Hille [6] mentioned earlier in connection with the operator \( z \, d/dz \). The terms in (2.8) are also defined for \( n = -1 \), and when \( \alpha = 0 \) the equation reduces to the Cauchy Integral Formula for derivatives. However, all the results given below require \( n > 0 \), and so we retain this assumption throughout.

We are now going to replace \( \zeta \) in (2.8) by an indexed variable \( \zeta_n \) so as to make the sequence \( G_n(\zeta_n, t) \) converge, as \( n \to \infty \). The choice of \( \zeta_n \) is suggested by the following lemma.

Lemma 2.1. For fixed \( m > 1 \), and \( p > 0 \), the sequence

\[
S_{mn}^{(p)} = \frac{C_{mn}^{(p)}}{C_n^{(p)}} \left[ \frac{C_n^{(p)}}{C_{2n}^{(p)}} \right]^{m-1} \quad (n = 1, 2, 3, \ldots)
\]

is convergent. Moreover \( C_{2n}^{(p)}/C_n^{(p)} \to 0, n \to \infty \).

Proof. If \( n = 1 \), we have

\[
S_{m1}^{(p)} = (m/2^{m-1}) < 1, \quad m > 1.
\]

Next, observe that \( S_{m,n+1}^{(p)} \) is obtained from \( S_{mn}^{(p)} \) by multiplying by the factor

\[
\frac{m + np}{1 + np} \left[ \frac{1 + np}{2 + np} \right]^{m-1}.
\]

We claim that this factor is at most 1, for that would be equivalent to

\[
1 + \frac{m - 1}{1 + np} \leq \left[ 1 + \frac{1}{1 + np} \right]^{m-1},
\]

which is true owing to the binomial theorem. So the terms \( S_{mn}^{(p)} \) satisfy \( 0 < S_{mn}^{(p)} < 1 \) and are monotone decreasing as \( n \to \infty \). The first conclusion follows. As for the second, note that

\[
\frac{C_{2n}^{(p)}}{C_n^{(p)}} = \prod_{k=1}^{n} \left[ 1 + \frac{1}{1 + (k-1)p} \right].
\]
and this is seen to diverge to \(\infty\) by elementary infinite product analysis. This completes the proof.

Define the auxiliary generating functions \(H_n(x, t)\) by

\[
H_n(x, t) = \sum_{m=1}^{\infty} \frac{C_m(p)}{C_n(p)} \left[ \frac{C_m(p)xt}{C_n(p)(1 - at)} \right]^{m-1} = \sum_{m=1}^{\infty} S_m^{(p)} \left[ \frac{(xt)}{(1 - at)} \right]^{m-1}.
\]

(2.9)

Taking note of (2.6), it is clear that

\[
H_n(x, t) = \left( \frac{1 - at}{C_n(p)} \right)^2 \frac{G_n \left( \frac{C_n(p)x}{C_n(p)}, t \right)}{C_n(p)},
\]

for \(|t| < \frac{1}{\alpha + \frac{C_n(p)}{C_n(p)} |x|}\), or for \(|x| < \frac{C_n(p)}{C_n(p)} \left( \frac{1}{|t|} - |\alpha| \right)\).

(2.10)

Then \(H_n(x, t)\) is analytic in each variable separately in the regions indicated by (2.10).

Recalling that the sequence \(S_m^{(p)}\) decreases monotonically with \(n\) to some non-negative limit \(S_m^{(p)} = 0 < S_m^{(p)} < S_m^{(p)} < 1\), let us define

\[
H(x, t) = \sum_{m=1}^{\infty} S_m^{(p)} \left[ \frac{(xt)}{(1 - at)} \right]^{m-1}.
\]

(2.11)

Because of its coefficients, (2.11) converges absolutely whenever (2.9) does. For each \(n\), (2.9) converges when the variables satisfy (2.10). Since \(H(x, t)\) does not depend on \(n\) and \(\left( \frac{C_1(p)}{C_n(p)} / \frac{C_2(p)}{C_n(p)} \right) \to 0, n \to \infty\), it follows that (2.11) converges for arbitrary \(x\) when \(t\) is fixed and \(|t| < |\alpha|^{-1}\), and for \(|t| < |\alpha|^{-1}\) when \(x\) is any fixed number. Therefore, \(H(x, t)\) is entire in \(x\) and analytic in \(t\) for \(|t| < |\alpha|^{-1}\). Recall that \(|\alpha| < 1\).

Note that we have

\[
H(x, t) = \lim_{n \to \infty} H_n(x, t),
\]

(2.12)

where the convergence is uniform on compact subsets of the admissible regions. Since \(S_{1n}^{(p)} = S_{2n}^{(p)} = 1, n > 1\), then

\[
H(x, t) = 1 + \frac{xt}{1 - at} + \ldots,
\]

so, in particular, \(H(x, t) \equiv 0\).

Let \(f, R\) and \(r\) be as in Theorem 2.1. Define

\[
I_n(x) = \frac{1}{2\pi i} \int_{|t| = r} \frac{f(-\frac{1}{t})H_n(x, t)}{(1 - at)^2} \, dt
\]

(2.13)

where \(x\) is such that (2.10) holds with \(|t| = r\). That is, (2.13) is defined, and \(I_n(x)\) is analytic for

\[
|x| < \frac{C_{2n}(p)}{C_{1n}(p)} \left( \frac{1}{r} - |\alpha| \right).
\]

(2.14)
Similarly, let

$$I(x) = \frac{1}{2\pi i} \int_{|t|=r} \frac{f(-t)}{(1 - at)^2} \, dt.$$ 

Then $I(x)$ is entire and, by (2.12), $I_n(x) \to I(x)$ uniformly on bounded sets in the plane. Since $H_n(0, t) = H(0, t) = 1$, we compute that

$$I_n(0) = I(0) = \frac{1}{2\pi i} \int_{|t|=r} \frac{f(-t)}{(1 - at)^2} \, dt = -f'(-\alpha).$$

More generally, the derivatives of $I(x)$ at $x = 0$ are given by

$$I^{(k)}(0) = \frac{S^{(p)}_{k+1}}{2\pi i} \int_{|t|=r} \frac{f(-t)}{(1 - at)^2} \left( \frac{t}{1 - at} \right)^k \, dt = -S^{(p)}_{k+1}f^{(k+1)}(-\alpha), \quad k = 0, 1, 2, \ldots.$$ 

A similar result holds for $I_n^{(k)}(0)$, with $S^{(p)}_{k+1}$ replaced by $S^{(p)}_{k+1,n}$. Therefore, neither $I_n(x)$ nor $I(x)$ vanishes identically unless $f$ is constant. For nonconstant $f$, we can find an integer $u = u(f)$ such that

$$I_u(x) = x^uJ_u(x), \quad I(x) = x^uJ(x), \quad (2.15)$$

where $I_n(0) \neq 0$ and $J(0) \neq 0$. Also, there will exist a constant $\gamma_f > 0$ such that $J(x) \neq 0$ for $|x| < \gamma_f$.

**Theorem 2.2.** Let $f, R$ and $r$ satisfy the hypothesis of Theorem 2.1, with $f$ nonconstant, and let $0 < \gamma < \gamma_f$. Then for all $n$ sufficiently large $(\theta^n f)(x)$ has no zero in the disc $|z + a| < \gamma C^{(p)}_{1n}/C^{(p)}_{2n}$.

**Proof.** On the contrary, suppose we could find a subsequence $z_{n_k}$ such that $(\theta^n f)(z_{n_k}) = 0$ and $z_{n_k} + \alpha = -x_{n_k} = -S^{(p)}_{k+1,n_k}/C^{(p)}_{2n_k}$, where $|x_{n_k}| < \gamma$, and where $n_k$ is large enough that (2.14) holds for all $k$. Combining (2.8), (2.10), (2.13) and (2.15), there follows

$$0 = (\theta^n f)(z_{n_k}) = -C^{(p)}_{1n_k}(z_{n_k} + \alpha)^1 + n_k x_{n_k} \frac{1}{C^{(p)}_{2n_k}} J_{n_k}(x_{n_k}), \quad (2.16)$$

and so $J_{n_k}(x_{n_k}) = 0$ for all $k$. Since $|x_{n_k}| < \gamma$, yet another subsequence of $\{x_{n_k}\}$ converges to a point $x_0$ such that $|x_0| < \gamma < \gamma_f$ and $J(x_0) = 0$. This contradiction proves the theorem.

**Remark.** It may be that $J(x)$ has no zeros, in which case $\gamma_f = \infty$. In this situation the discs $0 < |z + \alpha| < \gamma C^{(p)}_{1n}/C^{(p)}_{2n}$, for every $\gamma > 0$, are free of zeros of $(\theta^n f)(z)$ for all $n$ sufficiently large, depending on $\gamma$. Alternatively, $J(x)$ has zeros. If $J(x_0) = 0$, we determine by Hurwitz’s Theorem a sequence of points $x_n \to x_0$ such that $J_n(x_n) = 0$. If $z_n = -\alpha - x_n(C^{(p)}_{1n}/C^{(p)}_{2n})$, then $(\theta^n f)(z_n) = 0$ by (2.16), and we also have the asymptotic relation

$$\frac{C^{(p)}_{2n}}{C^{(p)}_{1n}}(z_n + \alpha) \sim |x_0| e^{i(\pi + \arg(x_0))}, \quad n \to \infty. \quad (2.17)$$
This is analogous to Theorem 35 of [7, pp. 172–173].

With regard to the asymptotic form of

\[ C_{2n}^{(p)} / C_n^{(p)} = \prod_{k=1}^{n} \left[ 1 + \frac{1}{1 + (k - 1)p} \right], \]

a straightforward analysis shows that

\[ e^{(1 + np)^{1/p}} > C_{2n}^{(p)} / C_n^{(p)} > \left( \frac{2 + p}{1 + p} \right)^{\log(1 + np)^{1/p}}, \quad p > 0. \]

As regards neighborhoods of points \( z \neq \alpha \), we cannot say as much. Let \( \beta \) satisfy \( |\beta| < R \), \( \beta \neq -\alpha \), and let \( w = T(z) = -[\gamma (z + \alpha)^p]^{-1} \), where the branch is chosen so as to be analytic at \( \beta \). The map is locally invertible, so there exists a function \( F(w) \) analytic at \( T(\beta) \) such that \( f(z) = F(T(z)) = F(w) \). By definition of \( T(z) \),

\[ (\theta^n f)(z) = (D^n F)(w), \quad n = 0, 1, 2, \ldots, \]

where \( D \) stands for ordinary differentiation. Apply the Whittaker-Gončarov theory to \( F(w) \) and translate the information over to the iterates \( (\theta^n f)(z) \). Unless \( f \) is a polynomial in \( (z + \alpha)^{-p} \), there exists a sequence of discs \( D_n \), shrinking with increasing \( n \) to \( z = \beta \), and a subsequence \( \{n_k\} \) such that \( (\theta^n f)(z) \) has no zero in punctured discs \( D_{n_k}, k = 1, 2, 3, \ldots \).

3. Applications. We consider zeros of successive derivatives of two classes of analytic functions, which correspond to taking \( \gamma = 1 \) and \( \gamma = 0 \) in Theorem 2.2.

Case I: \( \gamma = 1 \). Let \( F(w) \) be a function of the type considered by Boas [1] and Widder [7], that is,

\[ F(w) = b_1 w^{-1} + b_2 w^{-2} + \ldots, \quad \text{(nonconstant)} \]

analytic for \( |w| > R^{-1}, R > 1 \). Let \( f(z) = F(-1/z) \), and \((\theta^n f)(z) = z^{2^n}(z)\), so that \((\theta^n f)(z) = (D^n F)(w)\), \( n = 0, 1, 2, \ldots \). By Theorem 2.2, the regions \( 0 < |z| < \gamma(n + 1)^{-1} \) are eventually zero-free for \( \gamma < \gamma_f \). Thus for \( \gamma < \gamma_f \) and all \( n \) large, \( F^{(n)}(w) \) has no zero which satisfies \( \infty > |w| > \gamma^{-1}(n + 1) \), and this is the conclusion of Theorem A. Note that \( S_m^{(1)} = 1/(m - 1)! \), and so \( H(x, t) = \exp(xt) \) and

\[ I(x) = \frac{1}{2\pi i} \int_{|t| = r} F(t) e^{xt} \, dt. \]

That is, \( I(x) \) is the inverse Laplace transform of \( F \). Interpreting (2.17), any zero \( x_0 \neq 0 \) of \( I(x) \) gives rise to a sequence \( \{w_n\} \) of zeros of \( F^{(n)}(w) \) which asymptotically approach rays (see [7, Theorem 35])

\[ w_n \sim \frac{(n + 1)e^{-i \arg(x_0)}}{|x_0|}, \quad n \to \infty. \]

Case II: \( \gamma = 0 \). Let \( F(w) \) be a function of the form \( F(w) = f(e^w) \), where \( f(z) \) is analytic in \( |z| < R, R > 1 \). Then \( f(z) = F(\ln z) \), and \( F(w) \) is analytic in the half-plane \( \text{Re}(w) < \ln R \), periodic in the imaginary direction, and tends uniformly to a limit as \( \text{Re}(w) \to -\infty \). Define \( \theta \) by \( \theta = z \, d/dz \). Then with \( f(z) = F(\ln z) \), we have \((\theta^n f)(z) = (D^n f)(w)\). Theorem 2.2 asserts that constants \( \gamma > 0 \) exist for which the discs \( 0 < |z| < \gamma 2^{-n} \) contain no zeros of \( (\theta^n f)(z) \) for all \( n \) sufficiently large.
Equivalently, the region \( \text{Re}(\omega) < \ln \gamma - n \ln 2 \) is free of zeros of \( F^{(n)}(\omega) \). To the zeros of \( J(x) \) correspond horizontal lines, instead of rays from the origin. If \( J(x_0) = 0 \), then there exists a sequence \( \{\omega_n\} \) such that \( F^{(n)}(\omega_n) = 0 \) and

\[
\omega_n \sim \ln|x_0| - n \ln 2 + i(\pi + \arg(x_0)), \quad n \to \infty.
\]

REFERENCES


DEPARTMENT OF MATHEMATICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061