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A UNIQUENESS THEOREM FOR FIXED POINTS

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ABSTRACT. In a recent paper, R. Kellogg [3] showed that if F: $\bar{D} \to \bar{D}$ is a completely continuous map of the closure of a bounded, convex, open set D in a real Banach space X, $F \in C^1(D)$, 1 is not an eigenvalue of $F'(x)$ for $x \in D$, and $F(x) \neq x$ for $x \in \partial D$, then F has a unique fixed point in D. More recently, L. Talman [7] extended this result to k-set contractions when $k < 1$. The main result of this note is to show that, if the dimension of X is larger than one, the result of Kellogg and its extension by Talman remain valid provided that the set $\{x \in D$: 1 is an eigenvalue of $F'(x)\}$ has no accumulation points in D, the other assumptions remaining the same. This result is obtained as a corollary of a more general result which gives conditions under which the set of fixed points of F in D is connected.

In a recent paper, R. Kellogg [3] showed that if F: $\bar{D} \to \bar{D}$ is a completely continuous map of the closure of a bounded, convex, open set D in a real Banach space X, $F \in C^1(D)$, 1 is not an eigenvalue of $F'(x)$ for $x \in D$, and $F(x) \neq x$ for $x \in \partial D$, then F has a unique fixed point in D. The existence of a fixed point for F follows from the well-known Schauder Fixed Point Theorem, thus the result in [3] establishes the uniqueness of the fixed point. More recently, L. Talman [7] extended this result to k-set contractions when $k < 1$. Kellogg's theorem is also contained in a result of Berger [1, Theorem 5.4.7] where a simpler proof is given based on a more fundamental property of topological degree, namely the homotopy property. Both Kellogg and Talman base their proofs on the Leray-Schauder formula. The main result of this note is to show that, if the dimension of X is larger than one, the result of Kellogg and its extension by Talman remain valid provided that the set $\{x \in D$: 1 is an eigenvalue of $F'(x)\}$ has no accumulation points in D, the other assumptions remaining the same. The reader can easily construct counterexamples to this result when the dimension of X is one. We obtain this result as a corollary of a more general result which gives conditions under which the set of fixed points of F in D is connected. A result of Krasnoselskii and Perov (see Theorem 13.4 of [8]) gives a similar conclusion under different hypotheses.

Before proceeding, we recall some basic definitions. If $B \subseteq X$ is a bounded set, the set measure of noncompactness of B, $\alpha(B)$, is defined by $\alpha(B) = \inf\{\varepsilon > 0$: B has a finite cover by sets whose diameters do not exceed $\varepsilon\}$. A function F defined on a subset of X is a k-set-contraction if $\alpha(F(B)) < k\alpha(B)$ for all bounded subsets B of the domain of f. Since a set B is precompact if and only if $\alpha(B) = 0$, it follows that every completely continuous map is a 0-set-contraction. In what follows we

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make use of the topological degree for k-set-contractive perturbations of the identity developed by R. Nussbaum [5].

Main results. Let X be a real Banach space and let D be a bounded open connected subset of X. Let $F\colon \bar{D} \to X$ be a continuous k-set-contraction for some $k < 1$ which is continuously differentiable on D. Set $\psi = I - F$, where I denotes the identity map on X, and let $S = \{x \in \bar{D}\colon \psi(x) = 0\}$. Recall that ψ is a proper map [2], that is, $\psi^{-1}(K)$ is compact in $\bar{D}$ whenever K is compact in X, and that $\psi'(x)$ is a Fredholm map of index zero for all $x \in D$ [6]. Let

$$R = \{x \in D\colon \psi'(x)\colon X \to X \text{ is an isomorphism}\}.$$

Since $\psi'(x)$ is Fredholm of index zero,

$$R = \{x \in D\colon 1 \notin P\sigma(F'(x))\}$$

where $P\sigma(A)$ denotes the point spectrum of A. By the inverse function theorem, if $x \in R$ there exist open subsets U_x and V_x of X such that $x \in U_x \subset D$, $\psi(x) \in V_x$ and $\psi\colon U_x \to V_x$ is a homeomorphism. We can always assume that V_x is a ball and that ψ is injective on $\bar{U}_x$. Note that the set R is open.

LEMMA. $\deg(\psi, U_x, \psi(x))$ *is constant on components of* R.

PROOF. It is enough to prove that $\deg(\psi, U_x, \psi(x))$ is locally constant on R. Let $x_0 \in R$ and let $y \in U_{x_0}$. Let $\omega = U_{x_0} \cap U_y$. Then ω is open and $\deg(\psi, U_{x_0}, \psi(y)) = \deg(\psi, \omega, \psi(y)) = \deg(\psi, U_y, \psi(y))$ since the equation (for x) $\psi(x) = \psi(y)$ has no solutions in $\overline{U_{x_0} \setminus \omega}$ and $\overline{U_y \setminus \omega}$ because ψ is injective on $\bar{U}_{x_0}$ and $\bar{U}_y$. Therefore we need only prove that

$$\deg\big(\psi, U_{x_0}, \psi(x_0)\big) = \deg\big(\psi, U_{x_0}, \psi(y)\big).$$

In order to accomplish this, set $H(x, t) = \psi(x) - t\psi(y) - (1 - t)\psi(x_0)$ for $x \in \bar{U}_{x_0}$, $t \in [0, 1]$. Suppose $H(z, s) = 0$ for some $z \in \partial U_{x_0}$ and $s \in [0, 1]$. Then $\psi(z) = s\psi(y) + (1 - s)\psi(x_0) \in V_{x_0}$ since V_{x_0} is a ball and $\psi(y), \psi(x_0) \in V_{x_0}$. But $\psi\colon U_{x_0} \to V_{x_0}$ is a homeomorphism and so there exists $z_0 \in V_{x_0}$ such that $\psi(z_0) = \psi(z)$. This contradicts our assumption that ψ is injective on $\bar{U}_{x_0}$ and so H is an admissible homotopy. Thus

$$\begin{aligned}\deg\big(\psi, U_{x_0}, \psi(x_0)\big) &= \deg\big(H(\cdot, 0), U_{x_0}, 0\big) = \deg\big(H(\cdot, 1), U_{x_0}, 0\big)\\ &= \deg\big(\psi, U_{x_0}, \psi(y)\big).\end{aligned}$$

THEOREM. *Suppose that*

(1) $S \cap \partial D = \varnothing$ *and* $\deg(\psi, D, 0) = \pm 1$,

(2) R *is connected,*

(3) R *is dense.*

Then S *is nonempty and connected. If in addition,* $S \cap R \neq \varnothing$ *then* S *is a singleton.*

PROOF. By (1), $S \neq \varnothing$ and, since ψ is proper, S is compact. If S is not connected then there exist two nonempty compact sets S_1 and S_2 such that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$. Hence there exist open subsets Ω_1 and Ω_2 in D such that $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \varnothing$ and $S_i \subset \Omega_i$, $i = 1$ and 2. Thus

$$\pm 1 = \deg(\psi, D, 0) = \deg(\psi, \Omega_1, 0) + \deg(\psi, \Omega_2, 0)$$

and so we may assume that $\deg(\psi, \Omega_1, 0) \neq 0$. Let $x_2 \in S_2$. Then by (3) there exists a sequence $\{y_k\} \subseteq R$ such that $y_k \to x_2$ as $k \to \infty$. Thus there exist open sets U_k and V_k as above: $y_k \in U_k$, $\psi(y_k) \in V_k$ and ψ: $U_k \to V_k$ is a homeomorphism. We can, and do, assume that the diameter of V_k tends to zero as $k \to \infty$. By Smale's generalization of Sard's theorem [**4**], each V_k contains some points all of whose preimages in D are in R. For each k, we select such a point $p_k \in V_k$. Now, $p_k \to 0$ as $k \to \infty$ and so by (1) we may assume that $\psi^{-1}(p_k) \cap \partial D = \varnothing$. Since ψ is proper we can then conclude that $\psi^{-1}(p_k)$ contains a finite number, n_k, of elements and that

$$\deg(\psi, D, p_k) = \sum_{x \in \psi^{-1}(p_k)} \deg(\psi, U_x, p_k).$$

Now by the lemma, $\deg(\psi, U_x, \psi(x)) = l$ (a constant) on R and so $\deg(\psi, D, p_k) = ln_k$. Since $\deg(\psi, D, p_k) = \deg(\psi, D, 0) = \pm 1$ for all large k, this implies that $l = \pm 1$ and that $n_k = 1$. Letting $\psi(z_k) = p_k$, then we have $z_k \in U_k$ and since the diameter of $V_k \to 0$ as $k \to \infty$ the same holds for U_k so $z_k \to x_2$ as $k \to \infty$. It follows that $\psi^{-1}(p_k) \cap \overline{\Omega}_1 = \varnothing$ for all large k and hence $\deg(\psi, \Omega_1, p_k) = 0$. Since

$$\deg(\psi, \Omega_1, p_k) = \deg(\psi, \Omega_1, 0) \neq 0$$

for large k we have arrived at a contradiction. Thus S must be connected.

If in addition $S \cap R \neq \varnothing$ then S contains an isolated solution of $\psi(x) = 0$, hence S must be a singleton.

In general, S is not a singleton. Let $X = R^2$ and $D = (-1, 1) \times (-1, 1)$. Let F: $\overline{D} \to \overline{D}$ by $(x, y) \to (x(1 - y^2/2)g(x^2 - 1/4), -y)$ where g: $R \to R$ is a C^∞ function satisfying $g(t) = 1$ for $t \leq 0$, $g'(t) < 0$ for $t > 0$ and $g(3/4) = 1/2$. Then the hypotheses (1), (2) and (3) of the theorem are satisfied ($R^c = \{(x, 0): x \leq 1/2\}$ and $\deg(I - F, D, 0) = +1$) and $S = \{(x, 0): x \leq 1/2\}$.

COROLLARY 1. *Suppose that*
(a) $\overline{D}$ *is convex,*
(b) $F(\overline{D}) \subset \overline{D}$ *and* $S \cap \partial D = \varnothing$,
(c) $D \setminus R$ *has no accumulation points in* D,
(d) $\dim X > 1$.
Then S is a singleton.

PROOF. S is compact and nonempty since, by (b) we can assume that $\deg(\psi, D, 0) = +1$ (see [3]). If $\dim X > 1$ then, by (c), R is connected so the theorem applies. Hence S is connected and either S is a singleton or S has an accumulation point which must lie in S. Since $D \setminus R$ has no accumulation points $S \cap R \neq \varnothing$. The result now follows from the theorem.

The following is a trivial example of a situation where the corollary applies but Kellogg's theorem fails. Let $D = \{(x, y) \in \mathbf{R}^2: (x^2 + y^2)^{1/2} < 2/3\}$ and $F(x, y) = (x^2 - y^2, 2xy) = z^2$ where $z = x + iy$. Clearly F: $\overline{D} \to \overline{D}$ and $F(x, y) \neq (x, y)$ for $(x, y) \in \partial D$. The set $\{(x, y)$: 1 is an eigenvalue of $F'(x, y)\} = \{(1/2, 0)\} \in D$ and F has the unique fixed point $(0, 0)$.

Finally, we note that by approximation it is possible to remove the assumption that F is differentiable in the theorem.

COROLLARY 2. *Suppose that* $\psi = I - F$ *where* F *is a continuous k-set-contraction for some* $k < 1$ *on* $\bar{D}$, *with* D *a bounded open subset of* X. *Suppose that*

(a) $\deg(\psi, D, 0) = \pm 1$.

(b) *There exists a sequence of continuously differentiable k-set-contractions* F_n *on* $\bar{D}$ *such that* $\psi_n = I - F_n$ *converges to* ψ *uniformly on* $\bar{D}$.

(c) *For all large n*, $R_n = \{x \in D: \psi_n'(x)$ *is an isomorphism*$\}$ *is connected and dense in* D.

Then $S = \{x \in D: \psi(x) = 0\}$ *is connected.*

PROOF. Since the degree is continuous in its first and last arguments, there exist $r > 0$ and a positive integer N_1 such that

$$\deg(\psi_n - y, D, 0) = \deg(\psi, D, 0) = \pm 1$$

for all $n \geqslant N_1$ and $|y| \leqslant r$. Hence, for each $n \geqslant N_1$ and $|y| \leqslant r$, $S_n(y) = \{x \in D: \psi_n(x) = y\}$ is connected.

We now proceed exactly as in the theorem letting $S = S_1 \cup S_2$, $S_1 \cap S_2 = \varnothing$ and $S_i \subset \Omega_i$, $i = 1, 2$, where $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \varnothing$. As before

$$\pm 1 = \deg(\psi, \Omega_1, 0) + \deg(\psi, \Omega_2, 0)$$

so we may assume $\deg(\psi, \Omega_1, 0) \neq 0$. By choosing r smaller and N_1 larger if necessary we have that $\deg(\psi_n, \Omega_1, y)$ is defined and not equal to zero for $n \geqslant N_1$ and $|y| \leqslant r$. Now fix $n_1 \geqslant N_1$ so large that $\sup_{x \in \bar{D}} |\psi_{n_1}(x) - \psi(x)| < r$. Let $x_2 \in S_2$ and $y = \psi_{n_1}(x_2)$. Then $|y| < r$, so that $S_{n_1}(y)$ is connected. But $S_{n_1}(y) \cap \Omega_2 \neq \varnothing$ and $\deg(\psi_{n_1}, \Omega_1, y) \neq 0$ so $S_{n_1}(y) \cap \partial\Omega_1 \neq \varnothing$ and $S_n(y) \cap \partial\Omega_1 = \varnothing$. This contradicts that $S_{n_1}(y)$ is connected and proves the corollary.

In case F has a fixed point x_0 which can be shown to be isolated (e.g. F' exists at x_0 and is invertible) then the set S in Corollary 2 is a singleton.

REFERENCES

1. M. Berger, *Nonlinearity and functional analysis*, Academic Press, New York, 1977.
2. F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc. Sympos. Pure Math., Vol. 18, Amer. Math. Soc., Providence, R.I., 1976.
3. R. B. Kellogg, *Uniqueness in the Schauder fixed point theorem*, Proc. Amer. Math. Soc. **60** (1976), 207–210.
4. S. Smale, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.
5. R. D. Nussbaum, *The fixed point index and fixed point theorems for k-set-contractions*, Ph.D. thesis, Univ. of Chicago, 1969.
6. ______, *The radius of the essential spectrum*, Duke Math. J. **37** (1970), 473–478.
7. L. A. Talman, *A note on Kellogg's uniqueness theorem for fixed points*, Proc. Amer. Math. Soc. **69** (1978), 248–250.
8. E. Zeidler, *Vorlesungen über nichlinear Functionalanalysis*. I, *Fixpunktsatze*, Teubner, Leipzig, 1976.

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