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ON FUNDAMENTAL SEQUENCES OF TRANSLATES

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ABSTRACT. We find Müntz-type theorems for sequences of the form $\{f(t + c_n)\}$ or $\{\exp(-c_n t)f(t)\}$ on $[0, \infty)$.

Let R denote the set of real numbers, and R^+ the set of nonnegative real numbers. Given a function $f(t)$, by $F(x)$ we shall denote its Fourier transform; thus if $f(t)$ is integrable on R, $F(x) = \int_R \exp(xti)f(t)\,dt$.

A classical theorem of N. Wiener affirms that if $f(t)$ is in $L_2(R)$, and $T(f)$ is the linear span of the set of functions of the form $f(t + c)$, with c real, then $T(f)$ is dense in $L_2(R)$ if, and only if, $F(x) \neq 0$ a.e. in R (cf. [**1**, p. 100, Theorem 12]). Motivated by Wiener's theorem, we found in [**2**] necessary and sufficient conditions for sequences of the form $\{f(t + c_n)\}$ or $\{\exp(c_n ti)f(t)\}$ to be fundamental in $L_2(R)$. The purpose of this paper is to consider the same problem on R^+.

Let E denote one of the spaces $L_p(R^+)$ $(p \geq 1)$ or $C_0(R^+)$ (the space of functions continuous on R^+ that vanish at infinity, endowed with the uniform norm). Given a function $f(t)$, by $E(f)$ we shall denote the set of functions in E that vanish wherever $f(t)$ vanishes. For a given sequence $\{c_n\}$, let $T(\varepsilon)$ denote the series $\sum_{c_n \neq 0} |c_n|^{-\varepsilon}$. With this notation we can state our first result.

THEOREM 1. *Let $f(t)$ be a continuous function in E, not identically zero; let $\{c_n\}$ be a sequence of distinct complex numbers such that* $\operatorname{Re}(c_n) \geq \delta|c_n|$, *for some $\delta > 0$. Let $f_n(t) = f(t)\exp(-c_n t)$. Then $\{f_n(t)\}$ is fundamental in $E(f)$ if, and only if, $T(1)$ is divergent.*

Let S represent one of the spaces $L_p[0, 1]$ $(p \geq 1)$, or $C[0, 1]$ (the space of functions continuous on $[0, 1]$ endowed with the uniform norm), and let $S(g)$ be the set of functions in S that vanish wherever $g(x)$ vanishes. Making the change of variable $x = \exp(-t)$ we readily see that Theorem 1 is equivalent to

THEOREM 2. *Let $g(x)$ be a continuous function in S, not identically zero; let $\{c_n\}$ be a sequence of distinct complex numbers such that $c_0 = 0$, and* $\operatorname{Re}(c_n) \geq \delta|c_n|$, *for some $\delta > 0$. Then $\{g(x)x^{c_n}\}$ is fundamental in $S(g)$ if, and only if, $T(1)$ diverges.*

REMARK. For $f(t) = 1$, Theorem 1 reduces to a result of M. M. Crum [**3**]; it is clear, however, that Crum's theorem is equivalent to that of Müntz-Szász.

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For sequences of translates we have the following:

THEOREM 3. *Let* $\{c_n\}$ *be a sequence of distinct complex numbers. Assume* $f(z)$ *is a not identically zero entire function of exponential type* a, *square integrable on* R, *and for some* $\delta > 0$ *define* $q(z) = \exp(-\delta z)f(z)$, *and* $q_n(t) = q(t + c_n)$ (t *real*); *then for* $\{q_n(t)\}$ *to be fundamental in* E *it suffices that* $T(\varepsilon)$ *be divergent for some* $\varepsilon > 1$. *If in addition* $\mathrm{Re}(c_n) \geq 0$ *for all* n, *then the divergence of* $T(1)$ *is necessary for* $\{q_n(t)\}$ *to be fundamental in* E.

The gap between the necessary and sufficient conditions in Theorem 3 can be eliminated by considering a slightly different function, and imposing restrictions on the sequence $\{c_n\}$; thus we have:

THEOREM 4. *Let* $\{c_n\}$ *be a sequence of distinct complex numbers, bounded away from zero, with* $\mathrm{Im}(c_n) < 0$, $\mathrm{Re}(c_n) \geq 0$, *and such that* $\{\mathrm{Re}(c_n)\}$ *is a bounded sequence. Let* $f(z)$ *and* $q(z)$ *be as in Theorem* 3; *let* $h(z) = \exp(iaz)q(z)$, *and* $h_n(t) = h(t + c_n)$ (t *real*). *Then for* $\{h_n(t)\}$ *to be fundamental in* E, *it is necessary and sufficient that* $T(1)$ *be divergent.*

EXAMPLE. The function $f(z) = (\sin z)/z$ satisfies the hypotheses of Theorems 3 and 4.

Applying Theorem 4 with $a = \delta = 1$ and $\mathrm{Re}(c_n) = 1$, and eliminating redundant factors, we obtain the following:

COROLLARY. *Let* $f(z) \not\equiv 0$ *be a square-integrable entire function of exponential type* 1, *and let* $\{\lambda_n\}$ *be a strictly increasing sequence of strictly positive real numbers. Then the sequence* $\{\exp(-t)f(t - i\lambda_n)\}$ *is fundamental in* E *if, and only if,* $T(1)$ *is divergent.*

REMARK. The set of functions $f(z)$ that satisfy the hypotheses of Theorems 3 and 4 is characterized by a well-known theorem of Paley and Wiener (cf. [4, p. 13, Theorem X] or Boas [5, p. 103, 6.8.1]).

PROOF OF THEOREMS 1 AND 2. *Sufficiency*. We prove the sufficiency for Theorem 2. It suffices to assume that S is the set $C_0[0, 1]$, of continuous functions that vanish at 0. Assume $T(1)$ is divergent and let I be the set of points of $(0, 1]$ at which $g(x)$ does not vanish. By the theorem of Hahn-Banach we know it suffices to show that any linear functional on $C_0[0, 1]$ that annihilates the functions $g(x)x^{c_n}$, $n = 1, 2, 3, \ldots,$ is identically zero. Applying the Riesz representation theorem we readily conclude that the assertion is equivalent to showing that if μ is a (finite) complex-valued measure with support in I such that

$$\int_0^1 x^{c_n} g(x)\, d\mu(x) = 0, \qquad n = 1, 2, \ldots, \tag{1}$$

then $\mu = 0$. However (1) is equivalent to

$$\int_0^1 x^{c_n}\, d\nu(x) = 0, \qquad n = 1, 2, \ldots,$$

where $d\nu(x) = g(x)\, d\mu(x)$, and the theorem of Müntz-Szász implies that $\nu = 0$.

Hence $\nu(A) = \int_A g(x)\, d\mu(x) = 0$ for every measurable set A. Since $g(x) \neq 0$ on I, the conclusion follows.

Necessity. We prove the necessity for Theorem 1. Let $p > 1$ be given, and let q be such that $p^{-1} + q^{-1} = 1$. Since $f(t)$ is continuous and not identically zero, there is a closed interval $[\alpha, \beta]$, $\alpha > 0$, on which $f(t)$ does not vanish. Assume (1) is convergent; then, by a theorem of Luxemburg and Korevaar [6, p. 30, Theorem 5.2], there is a function $g(t)$ in $C^\infty(R)$ with supporting interval $[\alpha, \beta]$, such that the function $G(z) = \int_\alpha^\beta \exp(-izt)g(t)\, dt$ vanishes at the points $-ic_n$, i.e.

$$\int_\alpha^\beta \exp(-c_n t)g(t)\, dt = 0, \qquad n = 0, 1, 2, \ldots .$$

Thus, if $g^*(t) = g(t)/f(t)$, it is clear that $g^*(t)$ is in $L_q(\alpha, \beta)$, is not equivalent to zero, and

$$\int_\alpha^\beta f_n(t)g^*(t)\, dt = 0, \qquad n = 0, 1, 2, \ldots .$$

Thus $\{f_n(t)\}$ cannot be fundamental in $L_p[\alpha, \beta]$, whence the conclusion readily follows, bearing in mind that $C[0, 1]$ is dense in $L_p[0, 1]$ for every $p > 1$. Q.E.D.

Proof of Theorem 3. *Sufficiency*. It clearly suffices to assume that $E = C_0(R^+)$. Let $T(\varepsilon)$ be divergent for some $\varepsilon > 1$, and assume μ is a (finite) complex-valued measure with support in R^+, such that

$$\int_R q(t + c_n)\, d\mu(t) = 0, \qquad n = 0, 1, 2, \ldots .$$

Setting $d\nu(t) = \exp(-\delta t)\, d\mu(t)$, we readily conclude that

$$\int_R f(t + c_n)\, d\nu(t) = 0, \qquad n = 0, 1, 2, \ldots . \tag{3}$$

Let F and G be the Fourier transforms of f and ν respectively, and let

$$r(z) = (2\pi)\int_R f(t + z)\, d\nu(t). \tag{4}$$

We know from [5, p. 103, (6.8.2)] that, for real t, $|f(t + z)| < K \exp(a|y|)$, where y is the imaginary part of z, and K is a constant independent of t. Since ν is a finite measure, we conclude that (4) is defined for all complex z; thus, applying the theorems of Morera and Fubini, we readily see that (4) defines an entire function of exponential type. Since (from (3)), $r(z)$ vanishes at the points c_n, we conclude from [5, p. 17, 2.5.18] that $r(z)$ vanishes identically. Since (by Parseval's formula), $r(z)$ is the Fourier transform of $F(-t)G(t)$ (cf. Katznelson [7, p. 132], bearing in mind that we define Fourier transforms in terms of $\exp(zti)$, and not in terms of $\exp(-zti)$, and that the Fourier transform of $f(t + z)$ is $F(\xi)\exp(-\xi zi)$), we conclude that this product is equivalent to zero. Since $F(t)$ cannot be equivalent to zero (for this would imply that $f(z)$ vanishes identically), we see that $G(t)$ vanishes on a nondenumerable subset of the real line. However, since $d\nu(t) = \exp(-\delta t)\, d\mu(t)$, with μ bounded and with support in R^+, we readily see that G is holomorphic in $\operatorname{Im}(z) < \delta$. Since the real line is interior to this domain, we conclude that G vanishes identically thereon. Since the real and imaginary parts of ν are bounded, they can be represented as the difference of two finite and positive measures. Applying now Bochner's theorem on the uniqueness of the Fourier-Stieltjes transform (cf. Cotlar [8, p. 523, Theorem 3.1.9(c)]), we readily see that $\nu = 0$; hence (as in the proof of Theorems 1 and 2) also $\mu = 0$, and the conclusion follows.

Necessity. Let $f^*(z) = f(z)$ if $\mathrm{Re}(z) \geqslant 0$, and $f^*(z) = 0$ elsewhere, and let $q^*(z) = \exp(-\delta z)f^*(z)$. Since f is in $L_2(R)$, it is clear that also f^* is in $L_2(R)$, and we readily see that Q^* (the Fourier transform of q^*) is holomorphic in $\mathrm{Im}(z) < \delta$. Since $Q^*(t)$ is clearly not the zero function, there is an interval $[\alpha, \beta]$ on which $Q^*(-t)$ does not vanish. Assume that (1) converges. Then applying [6, p. 30, Theorem 5.2], we see there is a function $g(t)$ in $C^\infty(R)$, with supporting interval $[\alpha, \beta]$, such that the function $G(z) = \int_\alpha^\beta \exp(zti)g(t)\,dt$ vanishes at the points c_n. Clearly $G(z)$ is an entire function; moreover, since $Q^*(-t)$ does not vanish on $[\alpha, \beta]$, setting $g_1(t) = g(t)/Q^*(-t)$, on $[\alpha, \beta]$, and $g_1(t) = 0$ elsewhere, we have:

$$G(z) = \int_R \exp(zti)Q^*(-t)g_1(t)\,dt.$$

It is clear that $g_1(t)$ is in $C^\infty(R)$ and has bounded support; thus $g_1'(t)$ is bounded, and we infer that $g_1(t)$ satisfies a Lipschitz condition of order 1. Applying now a theorem of Bernstein (cf. [7, p. 32]) we conclude that $g_1(t)$ has an absolutely convergent Fourier series on any interval of the form $[\alpha - \eta, \beta + \eta]$, $\eta > 0$.

Making an obvious change of variable, and applying [5, p. 106, 6.8.11], it is easy to see that G_1 (the Fourier transform of g_1) is in $L_1(R)$. Since g_1 is in $L_1(R)$ we also know that G_1 is in $C_0(R)$ and is therefore bounded, and we readily infer that G_1 is in $L_q(R)$ for every $q \geqslant 1$. Applying Parseval's formula [4, p. 2, (1.09)] and noting that $2\pi g_1(t)$ is the Fourier transform of $G_1(-x)$, we see that for z real and positive,

$$2\pi G(z) = \int_{R^+} q^*(x + z)G_1(x)\,dx. \tag{5}$$

Since it is readily seen that the right-hand member of (5) is holomorphic in $\mathrm{Re}(z) > 0$ and continuous in $\mathrm{Re}(z) \geqslant 0$, (5) holds identically in the latter region.

Since $G(z)$ vanishes at the points c_n, we see from (5) that

$$\int_{R^+} q^*(x + c_n)G_1(x)\,dx = 0, \qquad n = 0, 1, 2, \ldots;$$

thus the sequence $\{q^*(x + c_n)\}$ cannot be fundamental in any $L_p(R^+)$. This implies that also $\{q_n\}$ cannot be fundamental thereon, whence the conclusion follows. Q.E.D.

Proof of Theorem 4. *Sufficiency*. Assume there is a (finite) complex-valued measure μ, with support in R^+, such that

$$\int_R h(t + c_n)\,d\mu(t) = 0, \qquad n = 0, 1, 2, \ldots. \tag{6}$$

Setting $d\nu(t) = \exp(-\delta t)\,d\mu(t)$, we readily see that (6) is equivalent to

$$\int_R \exp[a(t + c_n)i]f(t + c_n)\,d\nu(t) = 0, \qquad n = 0, 1, 2, \ldots. \tag{7}$$

Let F and G denote the Fourier transforms of f and ν respectively, and let

$$r(z) = \int_R \exp[a(t+z)i] f(t+z)\, d\nu(t).$$

Proceeding as in the proof of Theorem 3, we readily see that $r(z)$ is holomorphic and bounded in the region $\mathrm{Im}(z) < 0$. We also know from (7) that $r(z)$ vanishes at the points c_n. Since the function $v(z) = -i(1+z)/(1-z)$ transforms the interior of the unit circle conformally onto the region $\mathrm{Im}(z) < 0$, we see that the function $r[v(z)]$ is in the Hardy space H^∞, and vanishes at the points $\lambda_n = (c_n + i)/(c_n - i)$. Proceeding as in [**9**, p. 337], we conclude that $r(z)$ vanishes identically on $\mathrm{Im}(z) < 0$, and $r(z)$ being an entire function, this implies that it vanishes everywhere on the complex plane. Since (by Parseval's theorem) $2\pi r(z)$ is the Fourier transform of $F(a-t)G(t)$, we conclude that this product is equivalent to zero. Employing the same argument that was applied in the proof of Theorem 3, it is now easy to see that $\mu = 0$, whence the conclusion follows.

The proof of the necessity follows by applying the necessary part of Theorem 3 to $\exp(iaz)f(z)$ instead of to $f(z)$. Q.E.D.

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