AN APPLICATION OF YAU'S MAXIMUM PRINCIPLE TO CONFORMALLY FLAT SPACES

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Abstract. Results of M. Tani on compact conformally flat manifolds and of M. Okumura on compact hypersurfaces of Euclidean space are extended to complete spaces by an application of S.-T. Yau's "maximum principle".

1. Introduction. M. Tani [3] proved that a compact and orientable Riemannian manifold admitting a conformally flat metric of positive Ricci curvature and constant scalar curvature is a space form, that is, it is a constant curvature space. It is our purpose to extend this result to complete Riemannian manifolds with Ricci curvature bounded from below. This will be accomplished by employing a "maximum principle" due to S.-T. Yau. In fact, the following statement is obtained.

Theorem 1. Let $M$ be a $d$-dimensional, $d > 3$, complete, conformally flat Riemannian manifold whose Ricci curvature is bounded from below. If its scalar curvature $r$ is a positive constant and $\text{tr } Q^2 < r^2/(d - 1)$, then $M$ is a space form.

2. Definitions and notation. Let $(M, g)$ be a Riemannian manifold with metric $g$. The curvature transformation $R(X, Y), X, Y \in M_m, \text{ where } M_m$ is the tangent space at $m \in M, \text{ and } g$ are related by

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y],$$

where $\nabla$ is the Riemannian connection. In terms of a basis $X_1, \ldots, X_d$ of $M_m$, we set

$$R_{ijkh} = g(R(X_i, X_j)X_k, X_h), \quad R_{ij} = \text{tr}(X_k \rightarrow R(X_i, X_k)X_j),$$

$$t_{i_1 \ldots i_d} = t(X_{i_1}, \ldots, X_{i_d}), \quad \nabla_i t_{i_1 \ldots i_d} = (\nabla_{X_i}t)(X_{i_1}, \ldots, X_{i_d}).$$

We denote the scalar curvature by $r$, that is, $r = \text{tr } Q$, where $Q = (R_i^j)$ and $R_j^i = g^{ik}R_{jk}$. The manifold $(M, g)$ is conformally flat if $g$ is conformally related to a locally flat metric.

3. The Laplacian of $\text{tr } Q^2$. The following formula may be found in [1]:

$$\frac{1}{2} \Delta \text{tr } Q^2 = g^{ab} \nabla_a R_i^j \nabla_b R_{ij} + R \sum_{ij} g^{ab} \nabla_a (\nabla_b R_{ij} = \nabla_i R_{bj}) + \frac{1}{2} R_{ij} \nabla_i \nabla_j r + K,$$

(3.1)

where $\text{tr } Q^2$ is the square length of the Ricci tensor, and

$$K = R^{ij}(R_j^jR_{ik} + R^h_R_{ijk}).$$

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If \( r \) is a constant, the third term on the r. h. s. of (3.1) vanishes. If, moreover, \( M \) is conformally flat and \( d > 3 \), the second term on the right also vanishes (see [1]) and (3.1) reduces to

\[
\frac{1}{2} \Delta \text{tr} \, Q^2 = K + g(\nabla Q, \nabla Q).
\]

4. Proof of Theorem 1. Since \( M \) is conformally flat it can be shown that

\[
(d - 1)(d - 2)K = d(d - 1)\text{tr} \, Q^2 - r(2d - 1)\text{tr} \, Q^2 + r^3.
\]

Put \( S = Q - (r/d)I, I = \text{identity} \). Then, from \( \text{tr} \, S^2 > 0 \), we see that \( \text{tr} \, Q^2 > r^2/d \) with equality holding if and only if, \( M \) is an Einstein space. Since \( r \) is a constant, the Laplacian \( \Delta f^2 \) of the function \( f^2 = \text{tr} \, S^2, f > 0 \), satisfies \( \Delta f^2 = \Delta \text{tr} \, Q^2 \). Thus,

\[
\frac{1}{2} \Delta f^2 = K + g(\nabla Q, \nabla Q).
\]

Moreover,

\[
(d - 1)(d - 2)K = d(d - 1)\left( \text{tr} \, S^3 + \frac{3r}{d} f^2 + \frac{r^3}{d^2} \right) - r(2d - 1)\left( f^2 + \frac{r^2}{d} \right) + r^3.
\]

The following lemma may be found in [2].

**Lemma 1.** Let \( a_i, i = 1, \ldots, d, \) be real numbers with

\[
\sum_{i=1}^{d} a_i = 0, \quad \sum_{i=1}^{d} a_i^2 = k^2, \quad k = \text{const} > 0.
\]

Then,

\[
- \frac{d - 2}{\sqrt{d(d - 1)}} k^3 < \sum_{i=1}^{d} a_i^3 < \frac{d - 2}{\sqrt{d(d - 1)}} k^3.
\]

Applying Lemma 1 to the eigenvalues of \( S \), (4.2) yields the inequality

\[
(d - 1)K > f^2(r - \sqrt{d(d - 1)} f).
\]

We conclude from (4.1) that

\[
\frac{d - 1}{2} \Delta f^2 > f^2(r - \sqrt{d(d - 1)} f).
\]

**Lemma 2 (S.-T. Yau [4]).** Let \( M \) be a complete Riemannian manifold with Ricci curvature bounded below. Let \( u \) be a \( C^2 \) function with \( \sup u < \infty \). Then, there exists a sequence \( \{ p_r \} \) in \( M \) such that

\[
\lim_{r \to \infty} \| du(p_r) \| = 0, \quad \lim_{r \to \infty} (\Delta u)(p_r) < 0, \quad \lim_{r \to \infty} u(p_r) = \sup u.
\]

Applying Lemma 2, the inequality (4.3) gives rise to the inequality

\[
\lim_{r \to \infty} f^2(p_r)\{ r - \sqrt{d(d - 1)} f(p_r) \} < 0.
\]

Hence, either \( f^2 \equiv 0 \) or sup \( f > r/\sqrt{d(d - 1)} \), the latter implying \( \sup \text{tr} \, Q^2 > r^2/(d - 1) \). The former says that \( \text{tr} \, Q^2 = r^2/d \), so \( g \) is an Einstein metric. However, since \( g \) is conformally flat, it is a constant curvature metric.
The condition $\text{tr} \, Q^2 < r^2/(d - 1)$ is essential. For, if $M = M_1 \times N$, where $M_1$ has constant curvature and $N$ is 1-dimensional, then $M$ is conformally flat, its Ricci curvature is bounded below, $r$ is constant and $\text{tr} \, Q^2 = r^2/(d - 1)$.

In a similar manner, we obtain the following extension of a theorem of Okumura [2].

**Theorem 2.** Let $M$ be a $d$-dimensional complete connected hypersurface of $R^{d+1}$ with Ricci curvature bounded from below. If its mean curvature $\text{tr} \, H$ is constant and $\text{tr} \, H^2 < (\text{tr} \, H)^2/(d - 1)$, then $M$ is a totally umbilical hypersurface.

The inequality in Theorem 2 is the best possible as one sees by considering $M = S^{d-1} \times R$.

**References**


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