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SHRINKING DECOMPOSITIONS OF E^n WITH COUNTABLY MANY 1-DIMENSIONAL, STAR-LIKE EQUIVALENT NONDEGENERATE ELEMENTS

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ABSTRACT. It is shown that an upper semicontinuous decomposition of E^n ($n > 1$) with countably many 1-dimensional, star-like equivalent nondegenerate elements is shrinkable.

In this paper we address the problem of shrinking countable decompositions of E^n with star-like equivalent nondegenerate elements. Specifically, we prove that if G is an upper semicontinuous decomposition of E^n ($n > 1$) such that the collection H_G of nondegenerate elements is a countable collection of 1-dimensional, star-like equivalent sets, then $E^n/G \simeq E^n$. If the nondegenerate elements are tame (i.e. 1-LCC embedded), then, for $n \geqslant 5$, the theorem follows easily from recent results of R. D. Edwards **[Ed]**. However, in general, star-like sets need not be tame.

R. H. Bing **[Bi 1]** has shown that if H_G is a countable collection of star-like sets, then $E^n/G \simeq E^n$ ($n > 1$). R. J. Bean **[Be]** proved that $E^n/G \simeq E^n$ ($n > 1$) if H_G is a null sequence of star-like equivalent sets. The general case where H_G is a countable collection of star-like equivalent sets is not known for $n \geqslant 3$. (For $n = 1$ the result is trivial and for $n = 2$ it follows from classical results.) The special case when H_G is a countable collection of tame n-cells is of interest. (An n-cell in E^n is tame if it is ambiently homeomorphic to a standardly embedded cell.) A recent theorem of Starbird and Woodruff **[S-W]** states that $E^3/G \simeq E^3$ if H_G is a countable collection of tame 3-cells. The analogous theorem is not known for $n \geqslant 4$. R. H. Bing **[Bi 1]** has shown that $E^n/G \simeq E^n$ ($n > 1$) if H_G is a countable collection of tame arcs. The theorem presented here is proved using techniques similar to those used by Bing. The key is the 1-dimensionality of the (star-like equivalent) nondegenerate elements.

Let X be a nonempty compact set in E^n and $p \in X$. The set X is *star-like with respect to p* if each geometric ray emanating from p intersects X in a connected set. Equivalently, if $x \in X$, $x \neq p$, then the straight line segment determined by x and p is contained in X. Generally, X is *star-like* if it is star-like with respect to one of its points and *star-like equivalent* if it is ambiently homeomorphic to a star-like set.

MAIN THEOREM. *If G is an upper semicontinuous decomposition of E^n ($n > 1$) such that H_G is a countable collection of 1-dimensional, star-like equivalent sets, then $E^n/G \simeq E^n$.*

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The theorem follows from Lemma 2 below, where we show that the decomposition G is shrinkable. In this setting G is shrinkable if given an open set U containing the union of the nondegenerate elements and $\varepsilon > 0$, then there is a homeomorphism of E^n onto itself which is the identity on $E^n - U$ and such that the image of each nondegenerate element has diameter less than ε. The classical Bing Shrinkability Criterion (see **[Bi 1]**, **[Bi 2]**) allows us to conclude that the quotient map π: $E^n \to E_n/G$ is approximable by homeomorphisms. Using standard techniques for shrinking decompositions with countably many nondegenerate elements we need only show that a single nondegenerate element can be shrunk without "stretching" other nondegenerate elements. That is, it suffices to verify Lemma 1.

LEMMA 1. *If $g_0 \in H_G$, $\varepsilon > 0$ and W is a neighborhood of g_0 in E^n, then there is a homeomorphism h: $E^n \to E^n$ such that*

(a) *h is the identity on $E^n - W$,*

(b) diam $h(g_0) < \varepsilon$,

(c) *if $g \in H_G$, then either* diam $h(g) < \varepsilon$ *or $h(g) \subset N_\varepsilon(g)$, where $N_\varepsilon(g)$ denotes the ε-neighborhood of g in E^n.*

For $g_0 \in H_G$, let θ: $E^n \to E^n$ be a homeomorphism with $\theta(g_0)$ star-like. Assuming that $\overline{W}$ is compact and appealing to the uniform continuity of θ^{-1} restricted to $\theta(\overline{W})$, Lemma 1 is a consequence of the next result.

LEMMA 2. *If G is a monotone upper semicontinuous decomposition of E^n, $g_0 \in H_G$, g_0 is a 1-dimensional star-like set, $\varepsilon > 0$ and W is a neighborhood of g_0 in E^n, then there exists a homeomorphism h: $E^n \to E^n$ such that*

(1) *h is the identity on $E^n - W$,*

(2) diam $h(g_0) < \varepsilon$,

(3) *if $g \in H_G$, then either* diam $h(g) < \varepsilon$ *or $h(g) \subset N_\varepsilon(g)$.*

PROOF OF LEMMA 2. The homeomorphism h is constructed first to satisfy (1) and (2) using techniques found in **[Bi 1]**. This construction is outlined below. Condition (3) will follow from a careful replacement of the neighborhood W which relies heavily on the 1-dimensionality of g_0.

Let g_0 be star-like with respect to x_0. For $i \geq 1$ let C_i be the round ball of radius $i\varepsilon/4$ centered at x_0. Let m be the least positive integer such that $g_0 \subset \operatorname{int} C_m$. Since g_0 is star-like it has a neighborhood system, $\{D_i\}$, of n-cells which are ideally star-like with respect to x_0. That is, Bd D_i is a tame $(n-1)$-sphere and each geometric ray emanating from x_0 pierces Bd D_i in exactly one point (see **[Bi 1**, Lemma 3]). Using upper semicontinuity we choose from this system cells D_i, $1 \leq i \leq m$, such that

(a) $g_0 = D_0 \subset \operatorname{int} D_1 \subset D_1 \subset \cdots \subset \operatorname{int} D_m \subset D_m \subset C_m \cap W$,

(b) if $g \in H_G$ and $g \cap \operatorname{Bd} D_i \neq \varnothing$ then $g \cap D_{i-1} = \varnothing$, $1 \leq i \leq m$.

We define the action of h on each geometric ray R emanating from x_0. Let $y_0 = x_0$ and for $1 \leq i \leq m$ let $x_i = R \cap \operatorname{Bd} C_i$ and $y_i = R \cap \operatorname{Bd} D_i$. Let f be the map of $R \cap D_m$ onto $R \cap C_m$ such that $f(y_i) = x_i$, $0 \leq i \leq m$, and taking the segments $y_i y_{i+1}$ linearly onto the segments $x_i x_{i+1}$, $0 \leqslant i \leqslant m-1$. Define h to be the identity on $R - D_m$ and for $x \in R \cap D_m$ let $h(x)$ be the nearer to x_0 of $x, f(x)$. This is precisely the homeomorphism described in **[Bi 1**, Lemma 4].

To see that (1) is satisfied observe that h is the identity outside D_m and $D_m \subset W$. Clearly $h(D_1) \subset C_1$ and consequently diam $h(g_0) \leqslant$ diam $C_1 = \varepsilon/2$ and (2) is satisfied. One important feature of h should be isolated.

$$\text{If } x \in D_i - D_{i-1} \text{ and } h(x) \neq x, \text{ then } h(x) \in C_i - C_{i-1}. \qquad (*)$$

We now specify the restrictions on W that ensure (3). If $g_0 \subset C_1$ then no replacement for W is needed since $h = \text{id}$. Assume that $g_0 \not\subset C_1$. Since g_0 is 1-dimensional and star-like, $g_0 \cap \text{Bd}\, C_1$ is compact and 0-dimensional. Find a pairwise disjoint collection $U_1, \ldots, U_r$ of open subsets of Bd C_1 which cover $g_0 \cap \text{Bd}\, C_1$ and such that diam $\tilde{U}_j < \varepsilon/2$, $1 \leqslant j \leqslant r$, where $\tilde{U}_j$ denotes the radical projection of U_j from x_0 onto Bd C_m. Let V_j be the union of the straight line segments connecting x_0 with points of $\tilde{U}_j$; i.e. the geometric cone over $\tilde{U}_j$ from x_0. It follows that $V = (\bigcup_{j=1}^r V_j) \cup C_1$ is a neighborhood of g_0. We insist that W be contained in V. The crucial feature of W is that for $2 \leqslant i \leqslant m$ each component of $W \cap [C_i - C_{i-2}]$ has diameter less than ε.

To see that this restriction on W forces h to satisfy (3) suppose $g \in H_G$. Either there is an index i so that $g \subset D_i - D_{i-2}$ or $g \subset E^n - D_{m-1}$. In either case, if $K = \{x \in g | h(x) \neq x\}$, then $(*)$ shows that for some index i, $h(K) \subset C_i - C_{i-2}$. If $K = \varnothing$, then $h(g) = g \subset N_\varepsilon(g)$. If $K \neq \varnothing$ and A is a component of K, then $h(A)$ is contained in some component of $W \cap [C_i - C_{i-2}]$ and diam $h(A) < \varepsilon$. If $A = g$, then diam $g < \varepsilon$. Otherwise, since g is connected, there exists a point $x \in g \cap F_r(h(A))$ from which $h(A) \subset N_\varepsilon(g)$. It follows that $h(g) \subset N_\varepsilon(g)$. This establishes (3) and completes the proof of Lemma 2 and the theorem.

References

[Bi 1] R. H. Bing, *Upper semicontinuous decompositions of E^3*, Ann. of Math. (2) **65** (1957), 363–374.

[Bi 2] ______, *A homeomorphism between the 3-sphere and the sum of two solid horned spheres*, Ann. of Math. (2) **56** (1952), 354–362.

[Be] Ralph J. Bean, *Decompositions of E^3 with a null sequence of star-like equivalent non-degenerate elements are E^3*, Illinois J. Math. **11** (1967), 21–23.

[Ed] R. D. Edwards, *Approximating certain cell-like maps by homeomorphisms*, Notices Amer. Math. Soc. **24** (1977), A-649.

[S-W] Michael Starbird and Edythe P. Woodruff, *Decompositions of E^3 with countably many non-degenerate elements*, Proc. 1977 Georgia Topology Conf. (J. C. Cantrell, Ed.), Academic Press, New York, 1979.

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