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A NOTE ON EXTREMALLY DISCONNECTED SPACES

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ABSTRACT. A topological space X is said to be locally S-closed if each point of X has an open neighborhood which is an S-closed subspace of X. In this note it is shown that every locally S-closed weakly Hausdorff (or almost-regular) space is extremally disconnected.

1. Introduction. In 1976, T. Thompson **[11]** introduced the concept of S-closed spaces. Recently, the following sufficient conditions for an S-closed space to be extremally disconnected have been obtained.

THEOREM A (HERRMANN **[3]**). *An S-closed weakly Hausdorff space is extremally disconnected.*

THEOREM B (HERRMANN **[3]**). *An S-closed almost-regular space is extremally disconnected.*

THEOREM C (CAMERON **[1]**). *A maximal S-closed space is extremally disconnected.*

In **[7]**, the present author introduced the concept of locally S-closed spaces which is strictly weaker than that of S-closed spaces. The purpose of the present note is to show that the condition "*S-closed*" in the theorems above stated can be replaced by "*locally S-closed*".

2. Preliminaries. Let (X, τ) be a topological space and S a subset of X. The closure of S and the interior of S in (X, τ) are denoted by $\mathrm{Cl}_X(S)$ and $\mathrm{Int}_X(S)$, respectively. A subset S of X is said to be *regular open* (*regular closed*) if $\mathrm{Int}_X(\mathrm{Cl}_X(S)) = S$ (resp. $\mathrm{Cl}_X(\mathrm{Int}_X(S)) = S$). A topological space (X, τ) is said to be *extremally disconnected* if $\mathrm{Cl}_X(U) \in \tau$ for every $U \in \tau$. A subset S of (X, τ) is said to be *semiopen* **[4]** if there exists $U \in \tau$ such that $U \subset S \subset \mathrm{Cl}_X(U)$. The family of all semiopen sets in (X, τ) is denoted by $\mathrm{SO}(X, \tau)$.

DEFINITION 2.1. A topological space (X, τ) is said to be *S-closed* **[11]** if for every semiopen cover $\{U_\alpha | \alpha \in \nabla\}$ of X there exists a finite subfamily ∇_0 of ∇ such that $X = \bigcup \{\mathrm{Cl}_X(U_\alpha) | \alpha \in \nabla_0\}$.

A subset S of (X, τ) is said to be *S-closed* if it is S-closed as the subspace of (X, τ). A subset S of (X, τ) is said to be *S-closed relative to* τ **[6]** if for every cover $\{U_\alpha | \alpha \in \nabla\}$ of S by semiopen sets of (X, τ) there exists a finite subfamily ∇_0 of ∇ such that $S \subset \bigcup \{\mathrm{Cl}_X(U_\alpha) | \alpha \in \nabla_0\}$.

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Definition 2.2. A topological space (X, τ) is said to be *locally S-closed* [7] if each point of X has an open neighborhood which is an S-closed subspace of (X, τ).

Every S-closed space is locally S-closed. However, the converse is not true in general because an infinite discrete space is locally S-closed but not S-closed. The following lemmas shown in [7] will be used in the sequel.

Lemma 2.3. *Let A and B be subsets of a topological space (X, τ). If A is S-closed relative to τ and B is regular open, then $A \cap B$ is S-closed relative to τ.*

Lemma 2.4. *For a topological space (X, τ), the following are equivalent.*

(1) *(X, τ) is locally S-closed.*

(2) *For each $x \in X$, there exists $U \in \tau$ such that $x \in U$ and U is S-closed relative to τ.*

(3) *For each $x \in X$, there exists $U \in \tau$ such that $x \in U$ and $\mathrm{Int}_X(\mathrm{Cl}_X(U))$ is S-closed relative to τ.*

3. The results.

Definition 3.1. A topological space (X, τ) is said to be *weakly Hausdorff* [10] if each point of X is an intersection of regular closed sets of (X, τ).

Theorem 3.2. *If a topological space (X, τ) is locally S-closed and weakly Hausdorff, then it is extremally disconnected.*

Proof. Assume that (X, τ) is not extremally disconnected. Then, there exists a regular open set G of (X, τ) such that $\mathrm{Cl}_X(G) - G \neq \emptyset$ and $X - \mathrm{Cl}_X(G) \neq \emptyset$. Let $x \in \mathrm{Cl}_X(G) - G$. By Lemma 2.4, there exists an open neighborhood V of x such that V is S-closed relative to τ. Put $A = G \cap V$, then by Lemma 2.3, A is S-closed relative to τ. Since (X, τ) is weakly Hausdorff and $x \notin A$, for each $a \in A$ there exists a regular closed set $F(a)$ such that $x \notin F(a)$ and $a \in F(a)$. Since $F(a) \in \mathrm{SO}(X, \tau)$, there exists a finite subset A_0 of A such that $A \subset \bigcup \{F(a) | a \in A_0\}$. We have $A \subset V \cap \mathrm{Cl}_X(G) \subset \mathrm{Cl}_X(V \cap G) = \mathrm{Cl}_X(A)$. Therefore, $x \in \mathrm{Cl}_X(A) \subset \bigcup \{F(a) | a \in A_0\}$. On the other hand, $x \notin F(a)$ for any $a \in A_0$ and hence $x \notin \bigcup \{F(a) | a \in A_0\}$. This contradiction implies that (X, τ) is extremally disconnected.

Corollary 3.3. *A weakly Hausdorff space (X, τ) is S-closed if and only if it is locally S-closed and quasi H-closed.*

Definition 3.4. A topological space (X, τ) is said to be *almost-regular* [9] if for each $x \in X$ and each regular closed set F not containing x there exist disjoint open sets U and V of (X, τ) such that $x \in U$ and $F \subset V$.

Theorem 3.5. *If a topological space (X, τ) is locally S-closed and almost-regular, then it is extremally disconnected.*

Proof. Assume that (X, τ) is not extremally disconnected. Then there exists a regular open set G of (X, τ) such that $\mathrm{Cl}_X(G) - G \neq \emptyset$ and $X - \mathrm{Cl}_X(G) \neq \emptyset$. Let $x \in \mathrm{Cl}_X(G) - G$. Then, by Lemma 2.4, there exists a regular open neighborhood O of x which is S-closed relative to τ. Put $A = O \cap G$, then by Theorem 1.2 of [6] and Lemma 2.3, A is an S-closed subspace of (X, τ). Let $\mathfrak{V}(x)$ be the family of all neighborhoods at x and $\mathfrak{F} = \{V \cap A \mid V \in \mathfrak{V}(x)\}$. Then, $\mathfrak{F}$ is a filter base on A and hence, by Theorem 2 of **[11]**, $\mathfrak{F}$ s-accumulates to some point $a \in A$. Since (X, τ) is almost-regular and A is regular open, there exists $U \in \tau$ such that $a \in U \subset \mathrm{Cl}_X(U) \subset A$ [**9**, Theorem 2.2]. Since $x \notin A$, $(X - \mathrm{Cl}_X(U)) \cap A \in \mathfrak{F}$ and $a \in U \in \mathrm{SO}(A)$. Moreover, we have $[(X - \mathrm{Cl}_X(U)) \cap A] \cap \mathrm{Cl}_A(U) = \varnothing$ which contradicts that $\mathfrak{F}$ s-accumulates to $a \in A$. This shows that (X, τ) is extremally disconnected.

COROLLARY 3.6. *An almost-regular space (X, τ) is S-closed if and only if it is locally S-closed and quasi H-closed.*

DEFINITION 3.7. A locally S-closed space (X, τ) is said to be *maximal locally S-closed* if $\tau = \theta$ whenever a topological space (X, θ) is locally S-closed and θ is stronger than τ.

A function f: $(X, \tau) \to (Y, \theta)$ is said to be *irresolute* **[2]** if $f^{-1}(V) \in \mathrm{SO}(X, \tau)$ for every $V \in \mathrm{SO}(Y, \theta)$.

LEMMA 3.8. *Let f: $(X, \tau) \to (Y, \theta)$ be an irresolute function. If $G \in \tau$ and G is S-closed in (X, τ), then $f(G)$ is S-closed in (Y, θ).*

PROOF. Let f_G: $G \to f(G)$ be a function defined by $f_G(x) = f(x)$ for every $x \in G$, where G (resp. $f(G)$) is the subspace of (X, τ) (resp. (Y, θ)). We shall show that f_G is irresolute. For any $V_0 \in \mathrm{SO}(f(G))$, there exists $V \in \mathrm{SO}(Y, \theta)$ such that $V_0 = V \cap f(G)$ [**8**, Theorem 3.2]. Since f is irresolute and $G \in \tau$, $f^{-1}(V) \cap G \in \mathrm{SO}(X, \tau)$ and hence, by Theorem 1 of **[5]**, $f_G^{-1}(V_0) = f^{-1}(V) \cap G \in \mathrm{SO}(G)$. This shows that f_G is irresolute. Since G is S-closed, it follows from Theorem 3.5 of **[12]** that $f_G(G) = f(G)$ is S-closed.

THEOREM 3.9. *If a topological space (X, τ) is maximal locally S-closed, then it is extremally disconnected.*

PROOF. Assume that (X, τ) is not extremally disconnected. Then, there exists a regular closed set B of (X, τ) such that $B \notin \tau$. Put $\tau(B) = \{U \cup (V \cap B) \mid U, V \in \tau\}$, then $\tau(B)$ is a topology on X which is strictly stronger than τ. We shall show that $(X, \tau(B))$ is locally S-closed. Let $i_X : (X, \tau) \to (X, \tau(B))$ be the identity function. Then, it is obvious that i_X is open. For each $U \cup (V \cap B) \in \tau(B)$, $V \cap B \in \mathrm{SO}(X, \tau)$ and hence $U \cup (V \cap B) \in \mathrm{SO}(X, \tau)$. This shows that i_X is semicontinuous. Therefore, i_X is irresolute [**5**, Theorem 7]. Since (X, τ) is locally S-closed, for each $x \in X$ there exists an open neighborhood U of x in (X, τ) such that U is S-closed. By Lemma 3.8, $i_X(U)$ is S-closed in $(X, \tau(B))$. Moreover, $i_X(U)$ is an open neighborhood of x in $(X, \tau(B))$ because i_X is open. This shows that $(X, \tau(B))$ is locally S-closed. This contradicts that (X, τ) is maximal locally S-closed. Therefore, (X, τ) is extremally disconnected.

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