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A COMBINATORIAL PROOF OF SCHUR'S 1926 PARTITION THEOREM

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ABSTRACT. One of the partition theorems published by I. J. Schur in 1926 is an extension of the Rogers-Ramanujan identities to partitions with minimal difference three. This theorem of Schur is proved here by establishing a one-to-one correspondence between the two types of partitions counted.

Many classical partition identities state that for each positive integer n the partitions of n with parts restricted to certain residue classes are equinumerous with the partitions of n on which certain difference conditions are imposed. Most prominent among these are Euler's identity: partitions into odd parts are equinumerous with partitions into distinct parts; and the Rogers-Ramanujan identities: for $r = 1$ or 2, partitions into parts congruent to $\pm r \pmod 5$ are equinumerous with partitions into parts with minimal difference two and smallest part greater than or equal to r. In 1926, I. J. Schur discovered the partition theorem which is a natural extension [2].

SCHUR'S THEOREM. *Given any positive integer n, the partitions of n into parts congruent to* $\pm 1 \pmod 6$ *are equinumerous with the partitions of n into parts with minimal difference three and difference at least six between multiples of three.*

Many proofs of Euler's identity exist, including several which explicitly exhibit the correspondence between the two types of partitions counted (e.g., see [1, §19.4]). For the Rogers-Ramanujan identities, no such correspondence has been established. The purpose of this paper is to explicitly exhibit a one-to-one correspondence between the two types of partitions counted in Schur's Theorem. What makes the correspondence possible for Schur's identity and not for the Rogers-Ramanujan identities is that Schur's identity is a special case of the following partition theorem in which the parts restricted to certain residue classes are distinct.

SCHUR'S THEOREM (FULL GENERALITY). *Given positive integers r and m such that $r < m/2$, let $C_{r,m}(n)$ denote the numbers of partitions of n into distinct parts congruent to* $\pm r \pmod m$, *and let $D_{r,m}(n)$ denote the number of partitions of n into distinct parts congruent to* $0, \pm r \pmod m$ *with minimal difference m, minimal difference $2m$ between multiples of m. Then $C_{r,m}(n) = D_{r,m}(n)$ for all n.*

The special case given above is $r = 1$, $m = 3$. Clearly, $D_{1,3}(n)$ counts the partitions of n with minimal difference three, minimal difference six between

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multiples of three. To see that $C_{1,3}(n)$ also counts the partitions of n into parts congruent to $\pm 1 \pmod 6$, the reader is referred to the correspondence proof of Euler's identity given in Hardy and Wright [1, §19.4]. Following the correspondence given there, a one-to-one correspondence between odd parts with no multiples of three (i.e. parts $\equiv \pm 1 \pmod 6$) and distinct parts with no multiples of three (i.e. distinct parts $\equiv \pm 1 \pmod 3$) is easily established.

The correspondence given below will be for Schur's Theorem in its full generality.

Definition. Given positive integers r, m, n and k such that $r < m/2$, $n - mk(k-1)/2 \geqslant rk$, an *$r, m$ underlying partition for n and k*, or an *underlying partition*, is a partition of $n - mk(k-1)/2$ into exactly k parts, each of which is congruent to $0, \pm r \pmod m$, and such that multiples of m are distinct. (Example: 4, 6, 6, 11, 20, 40, 45 is a 1, 5 underlying partition for 237 and 7.)

Lemma 1. *There is a one-to-one correspondence between r, m underlying partitions for n and k and the partitions counted by $D_{r,m}(n)$ which have exactly k parts.*

Proof. Let $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$ be an r, m underlying partition for n and k and, for $1 \leqslant i \leqslant k$, define $b_i = a_i + m(i-1)$. Then $b_1 < b_2 < \cdots < b_k$ is a partition counted by $D_{r,m}(n)$, and this correspondence is uniquely reversible. (Example: 4, 6, 6, 11, 20, 40, 45 corresponds to 4, 11, 16, 26, 40, 65, 75.)

Definition. An ordering of the parts in an r, m underlying partition for n and k, say $(a_1, a_2, \ldots, a_k)$, is called an *Ω-ordering* if the following inequalities are satisfied for any i and j, $1 \leqslant i < j \leqslant k$.

(1) $a_i \equiv a_j \equiv 0 \pmod m \Rightarrow a_i < a_j$;

(2) $a_i \equiv \pm a_j \equiv \pm r \pmod m \Rightarrow a_i \leqslant a_j$;

(3)

$$\left.\begin{array}{l} a_i \equiv \pm r \pmod m \\ a_j \equiv 0 \pmod m \end{array}\right\} \Rightarrow a_i + mi \leqslant \left\lfloor \frac{a_j + mi}{2} \right\rfloor_r ;$$

(4)

$$\left.\begin{array}{l} a_i \equiv 0 \pmod m \\ a_j \equiv \pm r \pmod m \end{array}\right\} \Rightarrow \left\lfloor \frac{a_i + mi}{2} \right\rfloor_r < a_j + mi,$$

or equivalently,

$$\left\lceil \frac{a_i + m(i-1)}{2} \right\rceil_r < a_j + mi.$$

In this definition, $\lfloor A \rfloor_r$ (resp. $\lceil A \rceil_r$) is the greatest integer less than or equal to A (resp. least integer greater than or equal to A) and congruent to $\pm r \pmod m$.

Lemma 2. *Every underlying partition has a unique Ω-ordering.*

Proof. Given an r, m underlying partition for n and k, let $b_1 \leqslant b_2 \leqslant \cdots b_{k'}$ be the parts congruent to $\pm r \pmod m$ and let $c_1 < c_2 < \cdots < c_{k''}$ be the parts divisible by m. If $(a_1, a_2, \ldots, a_k)$ is an Ω-ordering of this partition, then a_1 equals either b_1 or c_1. By properties (3) and (4) of an Ω-ordering, if $b_1 + m \leq \lfloor (c_1 + m)/2 \rfloor_r$, then $a_1 = b_1$. If $b_1 + m > \lfloor (c_1 + m)/2 \rfloor_r$, then $a_1 = c_1$. Thus a_1 is uniquely determined.

We proceed inductively, and assume that $a_1, a_2, \ldots, a_{j-1}$ have been uniquely determined, and that we have used the parts $b_1, \ldots, b_{s-1}, c_1, \ldots, c_{t-1}$. Thus a_j is either b_s or c_t. Again by properties (3) and (4), if $b_s + mj \leq \lfloor (c_t + mj)/2 \rfloor_r$, then $a_j = b_s$. If $b_s + mj > \lfloor (c_t + mj)/2 \rfloor_r$, then $a_j = c_t$. Thus a_j is uniquely determined, and Lemma 2 is proved. (Example: The Ω-ordering of 4, 6, 6, 11, 20, 40, 45 is 4, 20, 6, 6, 40, 45, 11.)

DEFINITION. Given a partition, $a_1 < a_2 < \cdots < a_p$, counted by $C_{r,m}(n)$, we subdivide it, working from left to right, into blocks of at most two parts such that if $a_j + m > a_{j+1}$ and if (1) $j = 1$, or (2) $a_j \geq a_{j-1} + m$, or (3) a_{j-2} and a_{j-1} share a block, then a_j, a_{j+1} share a block. Otherwise, a_j inhabits a block by itself. The *order* of the partition is the number of such blocks. (Example: The partition 4, 11, 14, 16, 21, 29, 31, 34, 36, 41 counted by $C_{1,5}(237)$ is blocked as follows 4|11, 14|16|21|29, 31|34, 36|41. It has order 7.)

LEMMA 3. *There is a one-to-one correspondence between r, m underlying partitions for n and k and the partitions counted by $C_{r,m}(n)$ which have order k.*

PROOF. Let $(a_1, \ldots, a_k)$ be an r, m underlying partition for n and k with the Ω-ordering. Define a partition $b_1, \ldots, b_k$ of n by setting $b_i = a_i + m(i-1)$ for $1 \leq i \leq k$. Each b_i which is divisible by m is replaced by the pair $\lfloor b_i/2 \rfloor_r, \lceil b_i/2 \rceil_r$. From property (3) of the Ω-ordering, we have that if $i < j$, $b_i \equiv \pm r \pmod m$ and $b_j \equiv 0 \pmod m$, then $b_i + m \leq \lfloor b_j/2 \rfloor_r$. From property (4), we have that if $i < j$, $b_i \equiv 0 \pmod m$ and $b_j \equiv \pm r \pmod m$, then $\lceil b_i/2 \rceil_r < b_j$. Thus the resulting partition is one counted by $C_{r,m}(n)$ with order k. This procedure is uniquely reversed if parts sharing the same block are first added together, and then $m(i-1)$ is subtracted from the ith part as read from the left. (Example: 4, 20, 6, 6, 40, 45, 11 becomes 4, 25, 16, 21, 60, 70, 41 which becomes 4|11, 14|16|21|29, 31|34, 36|41.)

Lemmas 1 and 3, taken together, give the desired correspondence which establishes Schur's Theorem.

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REFERENCES

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