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A CHARACTERIZATION OF NORMAL OPERATORS USING THE HILBERT-SCHMIDT CLASS

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ABSTRACT. A bounded linear operator N on a Hilbert space H is normal if and only if $\|NX - XN\|_2 = \|N^*X - XN^*\|_2$ for every X in the Hilbert-Schmidt class.

Let H, $B(H)$ and C_2 denote a separable infinite dimensional Hilbert space, the set of all bounded linear operators acting on H and Hilbert-Schmidt class in $B(H)$, respectively.

It is well known that C_2 is a Hilbert space with the inner product $(X, Y) = \mathrm{Tr}(Y^*X)$ and the norm $\|X\|_2 = (X, X)^{1/2}$, where Tr is the natural trace on $B(H)$ [3].

For $N \in B(H)$, we define a bounded linear operator T on C_2 as follows:

$$TX = NX - XN \quad \text{for } X \in C_2.$$

Then a straightforward computation gives $T^*X = N^*X - XN^*$.

In [4], Weiss proved that if $N \in B(H)$ is a normal operator, $\|NX - XN\|_2 = \|N^*X - XN^*\|_2$ holds for every $X \in C_2$. Then, in [5] he proved that if N is normal, then $\|NX - XN\|_2 = \|N^*X - XN^*\|_2$ for every $X \in B(H)$. Also a characterization theorem for certain normal operators is given by Kamowitz [2]. He proved that $\|NX - XN\| = \|N^*X - XN^*\|$ for every $X \in B(H)$ if and only if N is normal and the spectrum of N lies on a circle or a straight line.

We prove another characterization of this kind, namely the converse of Weiss' theorem in [4]. We also give an alternate proof of his theorem. The central idea is to treat C_2 as a Hilbert space in its own right and to consider the operator T.

THEOREM. *$N \in B(H)$ is a normal operator if and only if $\|NX - XN\|_2 = \|N^*X - XN^*\|_2$ for every $X \in C_2$.*

PROOF. We see

$$T^*TX = N^*(NX - XN) - (NX - XN)N^*,$$
$$TT^*X = N(N^*X - XN^*) - (N^*X - XN^*)N.$$

Therefore, it follows $T^*TX = TT^*X$ if N is a normal operator. Hence we have $\|TX\|_2 = \|T^*X\|_2$.

Conversely, if the norm condition is satisfied, T is normal. Then we have $N^*NX + XNN^* = NN^*X + XN^*N$ for every $X \in C_2$. Therefore,

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$$(N^*N - NN^*)X = X(N^*N - NN^*) \quad \text{for every } X \in C_2.$$

It is well known and easy to prove that this implies that $N^*N - NN^* = a$ for some real number a. By [1, Problem 182], $a = 0$.

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