PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 2, June 1980

GLOBALLY STABLE COMPLETE MINIMAL SURFACES IN R^3

M. DO CARMO AND A. M. DA SILVEIRA

ABSTRACT. It is proved that a globally stable complete minimal surface in R^3 with finite total curvature is a plane.

The aim of this note is to give a proof of the following result.

THEOREM. *Let $x: M \to R^3$ be a complete minimal immersion of a two-dimensional, orientable, C^∞ manifold M. Assume that x is globally stable and that the total curvature of x is finite. Then $x(M) \subset R^3$ is a plane.*

Here *globally stable* means that each compact subdomain $W \subset M$ is a nondegenerate minimum for the area function of the induced metric for all variations that keep the boundary ∂W of W fixed. The result is clearly false for minimal surfaces in R^4 and we do not know whether it remains true without the finiteness of the total curvature.[1]

PROOF. Let $S^2 \subset R^3$ be the unit sphere with center at (0, 0, 0). We first prove that given a finite number of points $p_1, \ldots, p_k \in S^2$, there exists a domain $D \subset S^2$ such that $\lambda_1(D) = 2$, where λ_1 is the first eigenvalue of the Laplacian Δ in S^2, and D omits neighborhoods $U_i \subset S^2$ of p_i, $i = 1, \ldots, k$.

For each p_i, make a rotation of S^2 such that $p_i = (0, 0, 1)$ and define a function $u_i: S^2 \to R$ by the restriction to S^2 of

$$u_i(x_i, y_i, z_i) = 2 - z_i \log[(1 + z_i)/(1 - z_i)], \qquad (x_i, y_i, z_i) \in R^3.$$

It is easily checked that $\Delta u_i + 2u_i = 0$ (cf. [1, p. 519]) and that $\lim_{z_i \to \pm 1} u_i = -\infty$. Furthermore, u_i is positive in a ring-shaped domain D_i bounded by two parallels of S^2 and vanishes on the boundary ∂D_i. Thus $\lambda_1(D_i) = 2$. Now set $u = \sum_{i=1}^k u_i$ and define D as a connected component of the set $\{p \in S^2; u > 0\}$.

We claim that D is not empty. To see that we use the fact that of all spherical domains with the same area, the spherical cap has the smallest eigenvalue [3]. Since $D_1 \cap D_2 \neq \varnothing$, a connected component D_{12} of the set $\{p \in S^2; u_1 + u_2 > 0\}$ has eigenvalue 2. By the above, and the fact that the hemisphere has eigenvalue 2, we conclude that $A(D_{12}) > 2\pi$, where $A(\ )$ denotes the area of the enclosed domain. By the same token, $A(D_i) > 2\pi$; hence $D_{12} \cap D_3 \neq \varnothing$. An easy induction now shows that $A(D) > 2\pi$, as we claimed. Thus $\lambda_1(D) = 2$ and, since $\lim_{p \to p_i} u = -\infty$, D omits neighborhoods U_i of p_i, as we wished to prove.

Received by the editors May 1, 1979.

AMS (MOS) subject classifications (1970). Primary 53A10.

Key words and phrases. Minimal surface, stable, eigenvalues of the Laplacian.

[1]After the writing of this paper, this was shown to be true, independently, by Fischer-Colbrie, Schoen and do Carmo, Peng.

We need to use some results of R. Osserman [2]. The finiteness of the total curvature implies that M is conformally equivalent to a compact Riemann surface R minus a finite number of points $q_1, \ldots, q_k$. Furthermore, the Gauss map g: $M \to S^2$ can be analytically extended to a map $\bar{g}$: $R \to S^2$ and $\bar{g}$ is either constant or a branched covering of S^2; thus, if $\bar{g}$ is not constant, the (finitely many) points of S^2 that are omitted by $g(M)$ are all images of some q_i's under $\bar{g}$.

Now, assume that $x(M)$ is not a plane (i.e., $\bar{g}$ is not constant) and let $p_i = \bar{g}(q_i)$. By the previous construction, we can find a domain $D \subset S^2$ that omits neighborhoods U_i of p_i and such that $\lambda_1(D) = 2$. Let $\{q_1, \ldots, q_k, q_{k+1}, \ldots, q_n\} \subset R$ be the inverse image by $\bar{g}$ of the set $\{p_1, \ldots, p_k\}$. Choose neighborhoods $V_j \subset R$ of $q_j, j = 1, \ldots, n$, such that each $\overline{V}_j$ contains at most q_j as a branch point for $\bar{g}$, and $\bar{g}(V_j)$ is contained properly in some U_i. Set $M - \bigcup_j V_j = W$. Thus $\overline{W}$ is compact and, since $\bar{g}$ is a branched covering, $g(W) \not\subset D$. It follows that $\lambda_1(g(W)) < 2$. From Theorem 2.7 in [1], we conclude that W is unstable. This is a contradiction to the fact that x is globally stable, and completes the proof.

References

1. J. L. Barbosa and M. do Carmo, *On the size of a stable minimal surface in R^3*, Amer. J. Math. **98** (1976), 515–528.

2. R. Osserman, *Global properties of minimal surfaces in E^3 and E^n*, Ann. of Math. (2) **80** (1964), 340–364.

3. J. Peetre, *A generalization of Courant's nodal domain theorem*, Math. Scand. **5** (1957), 15–20.

Instituto Matematica Pura e Aplicada, Rio de Janeiro, Brasil

Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brasil

ABCDEFGHIJ–EB–89876543210