A NOTE ON STRONGLY $E$-REFLEXIVE INVERSE SEMIGROUPS

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Abstract. In contrast to the semilattice of groups case, an inverse semigroup $S$ which is the union of strongly $E$-reflexive inverse subsemigroups need not be strongly $E$-reflexive. If, however, the union is saturated with respect to the Green's relation $\mathcal{D}$, and in particular if the union is a disjoint one, then $S$ is indeed strongly $E$-reflexive. This is established by showing that $\mathcal{D}$-saturated inverse subsemigroups have certain pleasant properties. Finally, in contrast to the $E$-unitary case, it is shown that the class of strongly $E$-reflexive inverse semigroups is not closed under free inverse products.

The reader is referred to [1], [2] for the basic theory of inverse semigroups, including the theory of free inverse products. Recall from [4], [5] that an inverse semigroup $S$ is said to be strongly $E$-reflexive whenever $S$ is a semilattice of $E$-unitary inverse semigroups, or alternatively, whenever there exists a semilattice of groups congruence $\eta$ on $S$ such that only idempotents are linked to idempotents under $\eta$. In [4], [5] we studied this class of semigroups and showed that many of the properties of semilattices of groups and of $E$-unitary inverse semigroups generalise to this class, albeit sometimes in a weaker form. We continue this line of investigation here.

In what is by now a classic theorem, Clifford showed that an inverse semigroup which is a union of groups is a semilattice of groups. We ask to what extent this is true for strongly $E$-reflexive inverse semigroups. It is already known that a semilattice of strongly $E$-reflexive inverse semigroups is again strongly $E$-reflexive [5]. The following simple example shows that we cannot hope for a full generalisation of Clifford's theorem.

Consider the bisimple inverse $\omega$-semigroup $S(G, \alpha)$, where the endomorphism $\alpha$ of the group $G$ is not injective. As noted in [4, p. 341], $S(G, \alpha)$ is not strongly $E$-reflexive. However, using [1, Lemma 1.31], it is easily seen that $S(G, \alpha)$ is a union of its maximal subgroups and copies of the bicyclic semigroup, and these are all $E$-unitary.

The restriction we require will now be given, and the example just noted would seem to indicate that it is the weakest possible.

Let $S$ be an inverse semigroup with semilattice of idempotents $E$. Let $U$ be an inverse subsemigroup of $S$ which is $\mathcal{D}$-saturated in the sense that $x \mathcal{D} y \in U$ implies $x \in U$, where $\mathcal{D}$ denotes the usual Green's relation on $S$. The maximal group homomorphic image of $U$ is denoted by $\overline{U}$ with $\bar{u}$ denoting the image of $u$ ($u \in U$). Let $U' = \{ x \in S | x \geq u$ for some $u \in U \}$; note that $U'$ may equal $S$. 

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The first result shows that $U'$ has some pleasant properties.

**Proposition.** (i) $U'$ is an inverse subsemigroup of $S$ which contains $U$, and $xy \in U'$ implies $x \in U'$ and $y \in U'$.

(ii) The rule: $x \phi = \tilde{u}$ if $x \geq u \in U$ and $x \phi = 0$ otherwise, gives a well-defined homomorphism $\phi: S \rightarrow \bar{U}'$ such that $\phi|U$ is the canonical homomorphism onto $\bar{U}$.

**Proof.** (i) $xy \geq u \in U \Rightarrow xx^{-1} \geq xyy^{-1}x^{-1} \geq uu^{-1}x \Rightarrow x \geq uu^{-1}x \Rightarrow x \in U'$, since $U$ is $\mathfrak{D}$-saturated and $\mathfrak{D} \subseteq \mathfrak{D}$. Dually, $y \in U'$. The remainder of the result is easily proven.

(ii) Suppose $x \in U'$ with $x \geq u \in U$ and $x \geq v \in U$. Then $u = ex$, $v = fx$ where $e = uu^{-1} \in U \cap E$, $f = vv^{-1} \in U \cap E$. Hence efu = efv, and ef $\in E \cap U$, so that $\tilde{u} = \tilde{v}$. It is then almost immediate that $\phi$ is well-defined. The rest of the result involves a little routine calculation, using (i).

**Remark.** Taking $S$ to be a semilattice with more than two elements, we see that $U$ need not be an ideal of $U'$ in Proposition 1.

The proposition enables us to prove our main result.

**Theorem.** Let $S$ be a union of $\mathfrak{D}$-saturated strongly $E$-reflexive inverse subsemigroups $S_i$, $i \in I$. Then $S$ is strongly $E$-reflexive.

**Proof.** Each $S_i$ is a semilattice $\Lambda_i$ of $E$-unitary inverse semigroups $T_i^\lambda$, $\lambda \in \Lambda_i$. It is easily shown that each $T_i^\lambda$ is $\mathfrak{D}$-saturated in $S$. Hence we may suppose without loss of generality that each $S_i$ is $E$-unitary. For each $i \in I$, let $\phi_i: S \rightarrow \bar{S}_i^0$ be the homomorphism defined as in (ii) above, and let $T$ be the direct product of the $\bar{S}_i^0$. Then the $\phi_i$ induce a homomorphism $\phi: S \rightarrow T$ with $s\phi_i$ having $i$th component $s\phi_i$, $i \in I$. Now $S\phi$ is a semilattice of groups, since $T$ is. Suppose that $x\phi = e\phi$ for some $e \in E$, where $x \in S_i$, say. Then $x\phi_i$ is the identity element of $\bar{S}_i$, and since $S_i$ is $E$-unitary it follows that $x \in E$; whence the result.

**Corollary.** Let $S$ be a disjoint union of strongly $E$-reflexive inverse subsemigroups. Then $S$ itself is strongly $E$-reflexive.

**Proof.** Clearly each of the inverse subsemigroups in question is $\mathfrak{D}$-saturated in $S$.

**Remark.** The elementary theory of inverse semigroups shows that an inverse semigroup $S$ which is a union of groups is a disjoint union of its maximal subgroups $H_e$, $e \in E$, and that this is the $\mathfrak{D}$-decomposition of $S$. Hence $S$ is the union of the $\mathfrak{D}$-saturated $E$-unitary inverse subsemigroups $H_e$. It is easy to show that the homomorphism $\phi$ in the proof of the theorem is injective in this case. Hence $S$ is a subdirect product of the $H_e$ with zero added possibly. From this one can deduce, again by elementary means, that $S$ is a semilattice of groups with the multiplication defined by linking homomorphisms. Thus, modulo some elementary results, our theory restricts to Clifford's classic theorems.

Now let $E$ be the semilattice $\{e, f, g\}$ where $e \geq g, f \geq g$, and $e, f$ are incomparable. Let $S$ be the semilattice of groups $G_e \cup G_f \cup G_g$ where $G_e, G_g$ are trivial and $G_f$ is the cyclic 2-group; let $T$ be the semilattice of groups $H_e \cup H_f \cup H_g$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
where $H_g$ is the trivial group and $H_e, H_f$ are copies of the cyclic 2-group (the multiplications being defined in the obvious way). Consider the word $w = eabc$ in the free inverse product $P$ of $S$ and $T$, where $e$ is the identity element of $G_e$ and $a[b, c]$ is the non-identity element of $H_e[G_f, H_f]$. If $\psi$ is a semilattice of groups homomorphism on $P$, it is easily seen that $w\psi$ is an idempotent. On the other hand one can find a representation of $S$ and $T$ in $\delta_5$, the symmetric inverse semigroup on five symbols, in which the image of $w$ is not an idempotent. Hence $w$ is not an idempotent, so that $P$ is not strongly $E$-reflexive.

On the other hand McAlister [3] has shown that the free inverse product of two $E$-unitary inverse semigroups is again $E$-unitary.

References


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