TRACE POLYNOMIAL FOR TWO GENERATOR
SUBGROUPS OF SL(2, C)

CHARLES R. TRAINA

Abstract. If $G$ is a group generated by two $2 \times 2$ matrices $A$ and $B$ having
determinant $+1$, with entries from the complex field $\mathbb{C}$, it is known that the trace
of any word in $A$ and $B$, $W(A, B)$ is a polynomial with integral coefficients in the
two variables: $x = \text{trace}(A), y = \text{trace}(B), z = \text{trace}(AB)$, defined as
\[
\text{trace } W(A, B) = P(x, y, z),
\]
where $P$ is determined uniquely by the conjugacy class of $W(A, B)$.

The actual computation of this trace polynomial is not easily obtained. It is the
purpose of this paper to derive an explicit formula for this trace polynomial, and to
indicate some consequences of it.

1. Introduction. Fricke, in his work on automorphic functions [1] had shown the
following:

Let $G$ be a group generated by finitely many $2 \times 2$ matrices with entries from
the field $\mathbb{C}$ of complex numbers and with determinant $+1$. Then the trace of any
word in the generators is a polynomial with integral coefficients in finitely many
variables which are the traces of the generators and of finitely many of their
products. In particular, if $G$ has two generators $A$ and $B$, then the trace of any
word $W(A, B)$ is a polynomial in $x, y, z$ where
\[
x = \text{trace } A, \quad y = \text{trace } B, \quad z = \text{trace } AB.
\]

Horowitz [2] has shown that $x, y, z$ can be considered as independent variables
and that for every triplet of values $x_0, y_0, z_0$ there exists a pair of unimodular
matrices $A_0, B_0$ such that $A_0 = x_0, B_0 = y_0$ and $A_0B_0 = z_0$. In
addition, $A_0, B_0$ are, in general, i.e. whenever
\[
x_0^2 + y_0^2 + z_0^2 - x_0y_0z_0 - 4 \neq 0,
\]
uniquely determined up to conjugacy within the linear group $\text{SL}(2, \mathbb{C})$. Since the
group generated by two matrices $A, B$ is free of rank 2 unless $x, y, z$ satisfy at least
one of a countable number of algebraic equations with integral coefficients, it
follows that every conjugacy class of an element $W(a, b)$ in a free group on free
genators $a, b$ determines uniquely a polynomial $P(x, y, z)$ with integral coeffi-
cients if we define $P$ by
\[
\text{trace } W(A, B) = P(x, y, z).
\]
But Horowitz also showed that $P$ may not determine the conjugacy class of $W(a, b)$ uniquely, although, for a given $P$, there can exist only finitely many conjugacy classes represented by cyclically reduced words $W(a, b)$ such that (3) holds.

The proof of Fricke's theorem is based on the simple identities

$$\text{trace } UV + \text{trace } UV^{-1} = \text{trace } U \cdot \text{trace } V,$$
$$\text{trace } U^{-1} = \text{trace } U = \text{trace } WUW^{-1} \tag{4}$$

which is valid for any $W$ and any pair $U, V$ of elements of $\text{SL}(2, \mathbb{C})$. However, (4) does not lend itself easily to the actual explicit computation of $\text{trace } W(A, B)$ for an arbitrary $W$. On the other hand, applications of the results of Fricke and Horowitz to other group theoretical problems, for instance those treated by Rhee [6], Whittemore [7], Lyndon and Ullman [4] or Magnus [5] show that such explicit formulas would be very useful. It is the purpose of the present paper to derive and discuss formulas of this type.

Let $A$ and $B$ be two $2 \times 2$ matrices with entries from the complex field $\mathbb{C}$, having determinant $+1$.

Let $x = \text{trace}(A), y = \text{trace}(B), z = \text{trace}(AB)$.

Throughout the discussion, the following properties of the trace of matrices $U, V$ having determinant $+1$ will be used:

1. $\text{tr}(UV) = \text{tr}(VU),$
2. $\text{tr}(UV) = \text{tr}(U) \cdot \text{tr}(V) - \text{tr}(UV^{-1}),$
3. $\text{tr}(U^{-1}) = \text{tr}(U),$

and if $u = \text{tr}(U)$, then for any $n \in \mathbb{Z}$, $\text{tr}(U^n) = T_n(u)$, where $T_n(u)$ is the Chebyshev polynomial in the variable $t$, defined recursively as follows:

$$T_0(t) = 2, \quad T_1(t) = t,$$
$$T_n(t) = tT_{n-1}(t) - T_{n-2}(t), \quad T_n(2 \cos \varphi) = 2 \cos n\varphi.$$

We also need the polynomials $P_n(t)$ which satisfy the recurrence relations $P_n(t) = tP_{n-1}(t) - P_{n-2}(t), n \in \mathbb{Z}^+$, and the initial conditions $P_0(t) = 1, P_{-1}(t) = 0$. We have

$$P_n(2 \cos \varphi) = \sin(n + 1)\varphi / \sin \varphi.$$

The degree of $P_n(t)$ in $t$ is $n$ for $n > 2$ and $|n| - 2$ for $n < 1$. $P_{-1}(t)$ is identically zero and $P_n(t) = -P_{|n|-2}(t)$ for $n < 0$.

The polynomials $P_n(t)$ have as their generating function

$$P(t, u) = (u^2 - ut + 1)^{-1}$$

where $u$ is an arbitrary parameter.

The following property of the zeros of the polynomials $P_n(t), n > 2$, follows from examining their trigonometric representation, and is useful in establishing some of the results.

Let $n \in \mathbb{Z}^+, n > 2$. Then $P_n(t)$ has $n$ zeros, defined by

$$t(n, k) = 2 \cos(k\pi / (n + 1)), \quad k = 1, 2, \ldots, n,$$

in the interval $[-1, 1]$, with largest zero occurring at $t(n, 1) = 2 \cos(\pi / (n + 1))$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
These polynomials are related to the Chebyshev polynomials $T_n(t)$ as follows:

\[ T_n(t) = P_n(t) - P_{n-2}(t), \quad \text{for } n > 2. \]

The proofs of all results are omitted, since once the polynomials $P_n$ have been introduced, the results follow easily by induction, and properties (1), (2), (3) of the trace of matrices with determinant $+1$, previously stated.

2. The main theorem.

**Theorem 1.** Let $D = A^{u_1}B^{v_1} \cdots A^{u_n}B^{v_n}$, where $u_i, v_i \in \mathbb{Z} - \{0\}$ for $i = 1, 2, \ldots, n$, be a cyclically reduced $n$ syllable word. Then

\[ \text{tr}(D) = M_n(x,y)z^n + M_{n-1}(x,y)z^{n-1} + \cdots + M_0(x,y), \]

where the $M_i(x,y)$, $i = 0, 1, \ldots, n$, are sums of products of the polynomials $P_u(x)$, $P_v(y)$. Moreover, if $d_i$ represents the degree of $M_i(x,y)$, then

\[ \sum_{i=1}^{n} (|u_i| + |v_i|) - 2n < d_i < \sum_{i=1}^{n} (|u_i| + |v_i|) - 2n. \]

In particular, $M_n(x,y) = \prod_{i=1}^{n} P_{u_i-1}(x)P_{v_i-1}(y)$, and so $d_n = \sum_{i=1}^{n} (|u_i| + |v_i|) - 2n$. Thus, the degree of $\text{tr}(D)$ in $z$ equals the number of blocks $A^{u}B^{v}$.

3. Consequence of the main theorem.

**Corollary 1.** Cyclically reduced words $W_1$ and $W_2$ can have the same trace polynomial only if the absolute values of the exponents of the generators of $a$ in $W_2$ arise from those in $W_1$ by a permutation, and the same must be true for the exponents of $b$.

4. Generating function for traces of $n$ syllable words. Let $t_i, s_i$ be arbitrary scalar parameters, such that $0 < |t_i| < 1, 0 < |s_i| < 1$, for $i = 1, 2, 3, \ldots, l$. Consider the following matrix product

\[ \prod_{i=1}^{l} t_iA^{\eta_i}(I - t_iA^{\eta_i})^{-1}s_iB^{\eta_i}(I - s_iB^{\eta_i})^{-1}, \quad (1) \]

where $\epsilon_i, \eta_i = \pm 1$ for $i = 1, 2, \ldots, l$; and $I$ is the $2 \times 2$ identity matrix. By considering two different representations for this product based on the results

\[ (I - tM^e)^{-1} = \sum_{j=0}^{\infty} t^jM^{ej}, \quad (I - tM^e)^{-1} = (I - tM^{-e}) \sum_{j=0}^{\infty} P_j(u)t^j, \]

where $M$ is any $2 \times 2$ matrix with $\text{det}(M) = +1, u = \text{tr}(M), e = \pm 1$; one obtains the following expression involving the trace of an $n$ syllable word in $A$ and $B$:

\[ H(t, x, s, y)\text{tr}[ (A^{s_1} - t_1I)(B^{\eta_1} - s_1I) \cdots (A^{s_l} - t_lI)(B^{\eta_l} - s_lI) ] \]

\[ = \sum_{j, k=0}^{l} \prod_{i=1}^{l} t_i^{j_i}B^{\eta_i}t_i^{j_i+1}A^{s_i(l_i+1)}B^{\eta_i(k_i+1)}, \quad (2) \]

where

\[ H(t, x, s, y) = \sum_{j, k=0}^{l} \prod_{i=1}^{l} [P_{j_i}(x)P_{k_i}(y)t_i^{j_i+1}B^{\eta_i(k_i+1)}]. \]
If one considers fixed \( e_i, \eta_i = \pm 1; \ i = 1, 2, 3, \ldots, l; \) then from (2), one sees that \( \text{tr}(A^{e_1}B^{\eta_1} \cdots A^{e_l}B^{\eta_l}), \) where \( n_i, m_i = 1, 2, 3, \ldots \) will involve sums of products of the polynomials \( P_n(x), P_m(y) \) and the trace of the matrix product given by

\[
\text{tr}\left[ \prod_{i=1}^{l} (A^e - t_i I)(B^\eta - s_i I) \right].
\]

From this one sees that for \( \text{tr}(A^{e_1}B^{\eta_1} \cdots A^{e_l}B^{\eta_l}), \) where \( n_i, m_i = 1, 2, 3, \ldots \) for \( 1 < i < l, \) and \( e_i, \eta_i = \pm 1 \) for \( 1 < i < l \) are fixed, the trace is determined by the choice of the \( e_i, \eta_i. \) In fact, the signs of each pair \( (e_i, \eta_i); e_i \eta_i > 0 \) or \( e_i \eta_i < 0, \) will determine the degree of the trace polynomial in \( x, y, z, \) and the signs and degrees of the coefficients of the variable \( z. \)

Thus, in any rearrangement of the exponents, the signs of the products \( e_i \eta_i \) cannot be altered. Moreover, from the development of the generating function for an \( n \) syllable word, it can be seen that one cannot interchange exponents of the \( t_i \) and \( s_i, \) since this would give rise to a different term in the power series.

Hence, the only rearrangement of the exponents appearing in

\[
\text{tr}(A^{e_1}B^{\eta_1} \cdots A^{e_l}B^{\eta_l}), \text{ where } n_i, m_i \in \mathbb{Z}^+, 1 < i < l, e_i, \eta_i = \pm 1, 1 < i < l, \]

are fixed; which can occur without changing the trace, is one which can be done without altering the signs of the products \( e_i \eta_i. \) This leads to the following:

**Theorem 2.** In words of syllable length \( n: A^{a_1}B^{b_1} \cdots A^{a_k}B^{b_k}, \) we can apply a permutation \( \pi \) to the \( a_i \) without changing the traces if and only if this can be done in the case when all the \( a_i \) and all the \( b_i \) have value \( \pm 1. \)

**Acknowledgement.** The author wishes to thank Dr. Wilhelm Magnus for suggesting this topic and for his interest, invaluable help, and advice.

**Bibliography**


**Department of Mathematics and Computer Science, St. John's University, Jamaica, New York 11439**