AN INTEGRODIFFERENTIAL EQUATION

T. A. BURTON

Abstract. The vector equation

\[ x'(t) = A(t)x(t) + \int_0^t C(t, s)D(x(s))x(s) \, ds + F(t) \]

is considered in which \( A \) is not necessarily a stable matrix, but \( A(t) + G(t, t)D(0) \) is stable where \( G \) is an antiderivative of \( C \) with respect to \( t \). Stability and boundedness results are then obtained. We also point out that boundedness results of Levin for the scalar equation \( u'(t) = - \int_0^t a(t-s)g(u(s)) \, ds \) can be extended to a vector system \( x'(t) = - \int_0^t H(t, s)x(s) \, ds \).

1. Introduction. We consider the equation

\[ x'(t) = A(t)x(t) + \int_0^t C(t, s)D(x(s))x(s) \, ds + F(t) \]

in which \( A, C, \) and \( D \) are \( n \times n \) matrices, while \( x \) and \( F \) are \( n \)-vectors. In particular, \( A \) and \( F \) are continuous for \( 0 < t < \infty \), \( C \) is continuous for \( 0 < s < t < \infty \) and \( D \) is defined in a neighborhood of zero and continuous at \( x = 0 \).

Let \( G(t, s) \) be an \( n \times n \) matrix with

\[ \frac{\partial G(t, s)}{\partial t} = C(t, s) \]

and suppose that

\[ Q \overset{\text{def}}{=} A(t) - G(t, t)D(0) \]

commutes with its integral, while

\[ e^{\int_0^t Q(s) \, ds} < Me^{-\alpha(t-u)}, \quad 0 < u < t, \]

for some positive constants \( \alpha \) and \( M \). For example, (3) holds if \( A \) is constant and \( C \) is of convolution type, as \( Q \) would then be constant. Then (4) would hold if, in addition, the characteristic roots of \( Q \) all have negative real parts.

Under these conditions we obtain relations implying various stability results for (1). These results include cases in which \( A \) is constant with positive characteristic roots.

If \( D(x)x \) is locally Lipschitz and defined for all \( x \in \mathbb{R}^n \), then solutions of (1) are unique. If, in addition, a solution of (1) remains bounded, then it may be continued to \([0, \infty)\). However, our purpose here is not to examine existence and uniqueness, but only stability.

We assume throughout that any solution which remains in the domain of definition of \( D \) and which remains bounded may be continued to \( t = \infty \).

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2. Stability and boundedness. We first obtain an alternate form for (1) and an inequality for $|x|$.

**Lemma 1.** If $G$ satisfies (2), then (1) may be written as

$$x'(t) = \left[ A(t) - G(t, t)D(x(t)) \right] x(t) + \frac{d}{dt} f'G(t, s)D(x(s))x(s) \, ds + F(t).$$  

\hspace{1cm} (5)

**Lemma 2.** If $G$ satisfies (2) while the matrix $Q$ defined by (3) commutes with its integral, then the solution $x(t)$ of (1) with $x(0) = x_0$ satisfies

$$|x(t)| < |x_0|Me^{-\alpha t} + \int_0^t Me^{-\alpha(t-s)}|G(u, u)(D(0) - D(x(u)))x(u)| \, du$$

$$+ \int_0^t |G(t, s)D(x(s))x(s)| \, ds + \int_0^t Me^{-\alpha(t-s)}|F(u)| \, du$$

$$+ \int_0^t \int_s^t |Q(u)|Me^{-\alpha(t-s)}|G(u, s)D(x(s))x(s)| \, du \, ds.$$  

\hspace{1cm} (6)

**Proof.** Subtract $Qx$ from both sides of (5), multiply by $e^{-\int_0^t Q(s) \, ds}$, and group terms to obtain

$$\left( e^{-\int_0^t Q(s) \, ds}x(t) \right)' = e^{-\int_0^t Q(s) \, ds}\left\{ \left[ A(t) - G(t, t)D(x(t)) - Q \right] x(t) + \frac{d}{dt} f'G(t, s)D(x(s))x(s) \, ds \right\}.$$  

Using the definition of $Q$, integrating both sides from 0 to $t$, integrating the resulting next to last term by parts and interchanging the order of integration yields

$$e^{-\int_0^t Q(s) \, ds}x(t) = x_0 + \int_0^t e^{-\int_0^s Q(u) \, du}\left[ G(u, u)(D(0) - D(x(u))) \right] x(u) \, du$$

$$+ \int_0^t e^{-\int_0^s Q(u) \, du} \int_0^s G(t, s)D(x(s))x(s) \, ds$$

$$+ \int_0^t \int_s^t Q(u) e^{-\int_0^s Q(v) \, dv} G(u, s) du D(x(s))x(s) \, ds$$

$$+ \int_0^t e^{-\int_0^s Q(u) \, du} F(u) \, du.$$  

Now left multiply both sides by $e^{\int_0^t Q(s) \, ds}$, take norms and apply (4) to obtain (6).

**Remark 1.** Inequality (6) is the result from which stability conclusions may be drawn. We illustrate this by supposing that $G$ decreases exponentially.

**Lemma 3.** Let the conditions of Lemma 2 hold and suppose there are positive constants $J$ and $\beta$ with $|G(t, s)| < Je^{-\beta(s-t)}$ for $0 < s < t$. Suppose also that $\beta > \alpha$ and $Q$ is constant. If $P > |2>(0)|$, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that any solution $x(t)$ of (1) which satisfies $|x(t)| < \delta$ on $[0, \infty)$ also satisfies

$$|x(t)|e^{\alpha t} < M|x(0)| + \int_0^t \left\{ Me|G(u, u)| + \left[ |Q|MJP/ (\beta - \alpha) \right] 

\hspace{1cm} + \left[ JP - (|Q|MJP/ (\beta - \alpha)) \right] e^{(\alpha - \beta)(t-s)} \right\} |x(u)|e^{\alpha u} \, du$$

$$+ \int_0^t Me^{\alpha u}|F(u)| \, du.$$  

\hspace{1cm} (7)
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Proof. Multiply (6) by $e^{at}$ and note that by continuity of $D$ at $x = 0$, given $e > 0$ and $P > |D(0)|$ there exists $\delta > 0$ with $|D(0) - D(x)| < e$ and $|D(x)| < P$ if $|x| < \delta$. This information in (6) yields

$$|x(t)|e^{at} < M|x_0| + \int_0^t M e^{au}|G(u, u)|e^{au}|x(u)| du$$

$$+ \int_0^t M e^{au}|F(u)| du + \int_0^t J e^{(\alpha - \beta)(t - s)}|x(s)| e^{au} ds$$

$$+ \int_0^t \int_s^t Q |M| e^{(\alpha - \beta)(u - s)}|x(s)| e^{au} ds du.$$

Integration in the last term now yields (7).

When $G(s, s)$ is constant and $F(t) \equiv 0$, then (7) has the form

$$f(t) < a + \int_0^t \{be^{-\gamma(t-u)} + c\} f(u) du$$

with $a > 0, c > 0, \gamma > 0$ and $f > 0$.

Theorem 1. Let the conditions of Lemma 3 hold with $F(t) = 0$ and $\beta - \alpha < |Q|M$. If

$$\int_0^t \{Me|G(u, u)| + [Q|MJ|/(\beta - \alpha)]\} du < \alpha \varepsilon < \alpha t$$

for $t > 0$ and for some $\varepsilon > 0$, then the solution $x(t)$ of (1) with $x(0) = x_0$ tends to zero exponentially for $|x_0|$ small enough.

Theorem 1 follows from Gronwall’s inequality. It is weak as we need $|Q|MJ|/(\beta - \alpha)$ small and $|QM|/(\beta - \alpha) > 1$. Nevertheless, the possibility of $\beta - \alpha < |Q|M$ should not be neglected.

Theorem 2. Let the conditions of Lemma 3 hold and let $\beta - \alpha > |Q|M$. If $A$ and $G(t, t)$ are constant and if $J < a\beta/(\alpha Q|M + \alpha)$, then every solution of

$$y'(t) = Ay(t) + \int_0^t C(t, s)y(s) ds, \quad y(0) = y_0,$$

(1')

tends to zero exponentially as $t \to \infty$.

Proof. A review of the previous work will show that in the linear case we may take $\varepsilon = 0$ as $|D(0) - D(x)| = 0$ and we may take $P = 1$ and $\delta = \infty$. Our inequality (7) will become

$$|y(t)|e^{at} < |y(0)|M + \int_0^t \{[Q|MJ|/(\beta - \alpha)]$$

$$+J[1 - (\alpha Q|M/(\beta - \alpha))]^{(\alpha - \beta)(t-u)}\}|y(u)|e^{au} du$$

which we denote by (8) with all constants positive.

As $b > 0$, $f(t)$ is bounded above by the maximal solution of

$$g(t) = a + \int_0^t \{be^{-\gamma(t-u)} + c\} g(u) du$$

which is equivalent to

$$g'' + (\gamma - b - c)g' - \gamma cg = 0$$

(9)
with \( g(0) = a \) and \( g'(0) = a(b + c) \). The largest characteristic root is

\[
\lambda = \left[ b + c - \gamma + \left( (\gamma - b - c)^2 + 4 \gamma c \right)^{1/2} \right] / 2
\]

where \( b + c = J \), \( \gamma = \beta - \alpha \) and \( c = |Q|M J / (\beta - \alpha) \).

We recall from Lemma 3 that we need \( |y(t)| < \delta \) on \( [0, \infty) \) in order to maintain our inequalities. If we can make \( \lambda < \alpha \) then for sufficiently small \( |y_0| \) we will have \( |y(t)| < \delta \). At the same time, we will obtain the result that \( y(t) \to 0 \) exponentially for \( |y_0| \) small.

Now \( \lambda < \alpha \) if

\[
J + (\alpha - \beta) + \left( (\alpha + J - \beta)^2 + 4 |Q|M J \right)^{1/2} < 2\alpha,
\]

which is satisfied if

\[
J < \alpha \beta / [ |Q|M + \alpha ].
\]

**Corollary.** Let the conditions of Lemma 3 hold and let \( \beta - \alpha > |Q|M \). If \( A \) is constant and if \( J < \alpha \beta / [ |Q|M + \alpha ] \), then every solution of

\[
x'(t) = Ax(t) + \int_0^t C(t - s)x(s) \, ds + F(t), \quad x(0) = x_0, \quad (1)''
\]

satisfies

\[
x(t) = Z(t)x(0) + \int_0^t Z(t - s)F(s) \, ds
\]

where \( Z(t) \) tends to zero exponentially.

**Proof.** We have \( C(t, s) = C(t - s) \) so \( G(t, t) \) is constant. As \( A \) is also constant, according to [1] the solutions of \( (1)'' \) may be expressed as indicated where \( Z(t) \) is the \( n \times n \) matrix whose columns are solutions of \( (1)' \) with \( Z(0) = I \). By Theorem 2 the solutions of \( (1)' \) tend to zero exponentially.

**Example 1.** Consider the scalar equation

\[
x'(t) = Ax(t) + \int_0^t -ke^{-\beta(t-s)} [ \beta \cos(t - s) + \sin(t - s) ] x(s) \, ds.
\]

Then \( G(t, s) = ke^{-\beta(t-s)} \cos(t - s) \) so that \( G(t, t) = k, J = k, Q = A - k \) and \( e^{Qt} = e^{(A-k)t} \). Thus, \( M = 1 \) and we require \( A - k = -\alpha < 0 \).

Theorem 2 requires \( J < \alpha \beta / [ |Q|M + \alpha ] \) and \( \beta - \alpha > |Q|M \). These are satisfied if \( k < (k - A)\beta / (|k - A| + |k - A|) \) or \( k < \beta / 2 \) and if \( \beta + A - k > k - A \) or \( \beta / 2 > k - A \).

**Remark 2.** Let us examine Theorem 2 once more. If \( F(t) \equiv 0 \), if the conditions of Lemma 3 hold and if \( A \) and \( G(t, t) \) are constant, then we see that \( \epsilon \) can be made arbitrarily small and \( P \) can be made as close to 1 as we please. Thus, \( \beta - \alpha > |Q|M \) and \( J < \alpha \beta / [ |Q|M + \alpha ] \) should also be a sufficient condition for solutions of \( (1) \), starting sufficiently near zero, to approach zero exponentially.

**Remark 3.** In the convolution case, Grossman and Miller [3, p. 463] ask that the resolvent \( R(t) \) satisfy \( \int_0^\infty |R(t)| \, dt < \infty \) and \( \int_0^\infty |R'(t)| \, dt < \infty \) in order to obtain perturbation results. It is shown in [3, p. 552] that the columns of \( R(t) \) are solutions of \( (1) \) in case \( F(t) = 0 \) and \( C(t, s) = C(t - s) \). Thus, under the conditions of the corollary to Theorem 2 with \( F(t) = 0 \) we see that \( x(t) \) tends to zero exponentially.
If, in addition, $|C(t-s)| < M e^{-B(t-s)}$, a calculation will show that this implies $\int_0^\infty |x'(t)| \, dt < \infty$. Thus, under the above conditions we have

$$\int_0^\infty \left[ |R(t)| + |R'(t)| \right] \, dt < \infty.$$ 

Most authors require $A$ to be a stable matrix and treat the integral in (1) as a perturbation in order to obtain stability results for (1). There are notable exceptions.

Grossman and Miller [3, p. 552 and 558] show that in the linear convolution case the resolvent $R(t)$ satisfying $R'(t) = AR(t) + \int_0^t C(t-s)R(s) \, ds$, $R(0) = I$ with $\int_0^\infty |C(t)| \, dt < \infty$ will satisfy $\int_0^\infty |R(t)| \, dt < \infty$ if and only if

$$\det (s - A - \hat{C}(s)) \neq 0 \quad \text{for } \Re s > 0$$

where $\hat{C}(s)$ is the Laplace transform of $C$.

Grimmer and Seifert [2, p. 160] give a type of process that may transform (1) to an equation with $A$ a stable matrix.

Perhaps the very nicest result of all along the lines of $A$ being not stable is a very special case dealt with by Levin [4] who considers the scalar equation

$$u'(t) = -\int_0^t a(t-s)g(u(s)) \, ds \quad (11)$$

and proves

**Theorem (Levin).** Let $a(t)$ and $g(u)$ satisfy

(i) $a(t) \in C[0, \infty)$, $(-1)^k a^{(k)}(t) > 0$ for $0 < t < \infty$ and $k = 0, 1, 2, 3$, and $g(u) \in C(-\infty, \infty)$, $ug(u) > 0$ if $u \neq 0$,

(ii) $G(u) = \int_0^u g(s) \, ds \to \infty$ as $|u| \to \infty$.

If $a(t) \equiv a(0)$ and if $u(t)$ is any solution of (11) on $[0, \infty)$, then $\lim_{t \to \infty} u^{(j)}(t) = 0$, $j = 0, 1, 2$.

The result is special in that it is scalar, the kernel is of convolution type, the right side does not admit a term $Au$ with $A$ being possibly positive and the derivative conditions on $a(t)$ are severe indeed. On the other hand, the restrictions on $g$ are minimal and the proof involves perhaps the most clever construction of a Lyapunov functional to be found in the literature. In fact, it seems worthwhile to point out just how that Lyapunov functional can be extended to yield boundedness of a vector system. The fact that all scalars are symmetric and commute restricts the general statement.

**Theorem 3.** In the equation

$$x'(t) = -\int_0^t H(t, s)x(s) \, ds \quad (12)$$

we suppose that $H$ is an $n \times n$ matrix of functions continuous for $0 < s < t < \infty$, $H^T = H$, $H(t, 0)$ and $\partial H(t, s)/\partial s$ are continuous and positive semidefinite, while $(d/dt)H(t, 0)$ and $\partial^2 H(t, s)/\partial t \partial s$ are continuous and negative semidefinite. Then all solutions of (12) are bounded.
PROOF. Define a functional
\[ V(t, x(\cdot)) = x^T(t)x(t) + \int_t^t x^T(s) \, ds \, H(t, 0) \int_0^t x(s) \, ds \]
\[ + \int_0^t \left\{ \left[ \int_s^t x^T(q) \, dq \right] \left[ \frac{\partial H(t, s)}{\partial s} \right] \int_s^t x(q) \, dq \right\} \, ds \]
and differentiate \( V \) along a solution of (12) to obtain
\[ V'_{(12)}(t, x(\cdot)) = -\int_t^t x^T(s)H^T(t, s)x(s) \, ds \]
\[ - \int_0^t x^T(t)H(t, s)x(s) \, ds + x^T(t)H(t, 0) \int_0^t x(s) \, ds \]
\[ + \int_0^t x^T(s) \left\{ \left[ \frac{d}{dt}H(t, s) \right] \int_s^t x(s) \, ds + H(t, 0)x(t) \right\} \]
\[ + \int_0^t \left\{ x^T(t)\left[ \frac{\partial H(t, s)}{\partial s} \right] \int_s^t x(q) \, dq \right\} \, ds \]
\[ + \int_0^t \left[ \int_s^t x^T(q) \, dq \right] \left[ \frac{\partial^2 H(t, s)}{\partial t \partial s} \right] \int_s^t x(q) \, dq \]
\[ + \left[ \frac{\partial H(t, s)}{\partial s} \right] x(t) \right\} \, ds. \]

We integrate the last term in the last integral by parts. If we write it as \( \int_0^t u \, dv \)
then \( u = \int_s^t x^T(q) \, dq \) and \( dv = \left[ \frac{\partial H(t, s)}{\partial s} \right] x(t) \), so that \( v = H(t, s)x(t) - H(t, 0)x(t) \) and \( du = -x^T(t) \).

Likewise
\[ \int_0^t x^T(t)\left[ \frac{\partial H(t, s)}{\partial s} \right] \int_s^t x(q) \, dq \, ds = \int_0^t do \, u \]
may be integrated by parts. In this case, \( dv = x^T(t)\frac{\partial H(t, s)}{\partial s} \) and \( u = \int_s^t x(q) \, dq \)
so that \( v = x^T(t)H(t, s) - x^T(t)H(t, 0) \) and \( du = -x(s) \).

We then have
\[ V'_{(12)}(t, x(\cdot)) = -2 \int_0^t x^T(t)H(t, s)x(s) \, ds + x^T(t)H(t, 0) \int_0^t x(s) \, ds \]
\[ + \int_0^t x^T(s) \left[ \frac{dH(t, 0)}{dt} \right] \int_0^t x(s) \, ds \]
\[ + \int_0^t x^T(s)H(t, 0)x(t) - x^T(t)H(t, 0) \int_0^t x(q) \, dq \]
\[ + x^T(t)H(t, 0) \int_0^t x(q) \, dq \]
\[ + \int_0^t \left[ x^T(t)H(t, s) - x^T(t)H(t, 0) \right] x(s) \, ds \]
\[ + \int_0^t \left[ \int_s^t x^T(q) \, dq \right] \left[ \frac{\partial^2 H(t, s)}{\partial t \partial s} \right] \int_s^t x(q) \, dq \, ds \]
\[ + \int_0^t x^T(s) \left[ H(t, s)x(t) - H(t, 0)x(t) \right] \, ds \]
\[ = \int_0^t x^T(s) \left[ \frac{dH(t, 0)}{dt} \right] \int_0^t x(s) \, ds \]
\[ + \int_0^t \left[ \int_s^t x^T(q) \, dq \right] \left[ \frac{\partial^2 H(t, s)}{\partial t \partial s} \right] \int_s^t x(q) \, dq \, ds < 0. \]
As $V(t, x(\cdot)) > x^T(t)x(t)$ and $V'_{(12)} < 0$, all solutions are bounded.

**Example 2.** If

$$H(t, s) = \begin{pmatrix} \frac{c_1}{(t - s + 1)} & \exp - (2t - s) \\ \exp - (2t - s) & \frac{c_2}{(t - s + 2)^2} \end{pmatrix}$$

where $c_1 > 0$, $c_2 > 0$, and

$$c_1c_2/(t - s + 1)(t - s + 2)^4 > 4e^{-2(2t-s)}$$

for $0 < s < t < \infty$, then the conditions of Theorem 3 hold.

**Added in proof.** J. J. Levin has a scalar version of Theorem 3 in J. Differential Equations 4 (1968), 176–186.

**References**