ON THE REFLEXIVITY OF $C_0(N)$ CONTRACTIONS

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Abstract. Let $T$ be a $C_0(N)$ contraction on a separable Hilbert space and let
$J = S(\varphi_1) \oplus S(\varphi_2) \oplus \cdots \oplus S(\varphi_k)$ be its Jordan model, where $\varphi_1, \varphi_2, \ldots, \varphi_k$ are
inner functions satisfying $\varphi_j|\varphi_{j-1}$ for $j = 2, 3, \ldots, k$, and $S(\varphi_j)$ denotes the com-
pression of the shift on $H^2 \ominus \varphi_j H^2, j = 1, 2, \ldots, k$. In this note we show that $T$ is
reflexive if and only if $S(\varphi_1/\varphi_2)$ is.

In this note we only consider bounded linear operators defined on complex,
separable Hilbert spaces. For each operator $T$, let $\{T\}', \{T\}''$ and $\text{Alg } T$ denote
the commutant, double commutant and the weakly closed algebra generated by $T$
and $I$, respectively. Let $\text{Lat } T$ denote the lattice of invariant subspaces of $T$ and
$\text{Alg Lat } T$ denote the (weakly closed) algebra of operators which leave all the
subspaces in $\text{Lat } T$ invariant. Recall that $T$ is reflexive if and only if $\text{Alg Lat } T =$
$\text{Alg } T$. In [1] Deddens and Fillmore characterized reflexive operators on finite-di-
mensional spaces in terms of their Jordan canonical forms. Now we generalize their
result to $C_0(N)$ contractions. More specifically, we prove the following

Theorem 1. If $T$ is a $C_0(N)$ contraction and $J = S(\varphi_1) \oplus S(\varphi_2) \oplus \cdots \oplus S(\varphi_k)$ is
its Jordan model, then $T$ is reflexive if and only if $S(\varphi_1/\varphi_2)$ is.

A contraction $T (|| T || < 1)$ on a Hilbert space is of class $C_0(N)$ for some integer
$N > 1$ if there exists an inner function $\varphi$ such that $\varphi(T) = 0$ and the defect indices
of $T$, $d_T \equiv \text{rank}(I - T^* T)^{1/2}$ and $d_T^* \equiv \text{rank}(I - TT^*)^{1/2}$, are both equal to some
$M < N$. A $C_0(N)$ contraction is unitarily equivalent to the operator $T$ defined on
$H = H_N^2 \ominus \Theta_T H_N^2$ by $T f = P(e^{i}f)$ for $f \in H$, where $H_N^2$ denotes the standard
Hardy space of $C^N$-valued functions defined on the unit circle, $\Theta_T$ is the charac-
teristic function of $T$, and $P$ denotes the (orthogonal) projection from $H_N^2$ onto $H$ (cf.
[5, Chapter VII]). Two operators $T_1, T_2$ are quasi-similar if there exist one-to-one
operators $X$ and $Y$ with dense ranges (called quasi-affinities) such that $X T_1 = T_2 X$
and $Y T_2 = T_1 Y$. A $C_0(N)$ contraction is quasi-similar to a uniquely determined
Jordan operator (called its Jordan model) $J = S(\varphi_1) \oplus S(\varphi_2) \oplus \cdots \oplus S(\varphi_k)$,
where $\varphi_1, \varphi_2, \ldots, \varphi_k$ are inner functions satisfying $\varphi_j|\varphi_{j-1}, j = 2, 3, \ldots, k$, and
$S(\varphi_j)$ denotes the operator defined on $H^2 \ominus \varphi_j H^2$ by $S(\varphi_j)f = P(e^{i}f)$ for $f \in H^2$
$\ominus \varphi_j H^2, P_j$ being the (orthogonal) projection from $H^2$ onto $H^2 \ominus \varphi_j H^2, j =$
$1, 2, \ldots, k$ (cf. [4]). For $\xi$ and $\eta$ in $H^\infty, \xi \wedge \eta = 1$ denotes that $\xi$ and $\eta$ have no
nontrivial common inner divisor.

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We start the proof of Theorem 1 by showing that for $C_0(N)$ contractions, the property of reflexivity is preserved under quasi-similarities. This generalizes Corollary 4.5 in [7].

**Theorem 2.** Let $T_1$ and $T_2$ be $C_0(N)$ contractions on $H_1$ and $H_2$, respectively. Assume that $T_1$ is quasi-similar to $T_2$. Then $T_1$ is reflexive if and only if $T_2$ is.

**Proof.** We may assume that $T_1$ and $T_2$ are defined on $H_1 = H_1^* \ominus \Theta_1 H_N^*$ and $H_2 = H_2^* \ominus \Theta_2 H_N^*$ by $T_1 f = P_1(e^{t} f)$ and $T_2 g = P_2(e^{t} g)$, respectively, where $f \in H_1$ and $g \in H_2$. Since $T_1$ and $T_2$ are quasi-similar to each other, there exist bounded analytic functions $\Phi$ and $\Psi$ such that $\Phi \Theta_1 = \Theta_2 \Psi$ and $(\det \Phi)(\det \Psi) \wedge (\det \Theta_1)(\det \Theta_2) = 1$ (cf. [3] and [2]). Let $\Phi^t$ denote the algebraic adjoint of $\Phi$. It can be easily verified that the operators $X : H_1 \rightarrow H_2$ and $Y : H_2 \rightarrow H_1$ defined by $X f = P_2(e^t f)$ for $f \in H_1$ and $Y g = P_1((\det \Psi)\Phi^t g)$ for $g \in H_2$ implement the quasi-affinities intertwining $T_1$ and $T_2$ (cf. [2, Theorem 2]). Moreover, we have $YX = \eta(T_1)$ and $XY = \eta(T_2)$, where $\eta = (\det \Phi)(\det \Psi)$. Let $m_1$ and $m_2$ denote the minimal functions of $T_1$ and $T_2$, respectively. From the quasi-similarity of $T_1$ and $T_2$ we have $m_1 = m_2$.

Assume that $T_1$ is reflexive. Let $S \in \text{Alg Lat } T_2$ and $K \in \text{Lat } T_1$. Then $YSXK \subseteq \overline{YSXK} = \overline{\eta(T_1)K} K \wedge (\det \Theta_1) = 1$ implies that $\eta \wedge m_1 = 1$ (cf. [5, Theorem VI.5.2]). In particular, $\eta$ and the minimal function of $T_1|K$ have no nontrivial common inner divisor. Thus $\eta(T_1|K)$ is a quasi-affinity (cf. [7, Theorem 2.3]) and therefore $\overline{\eta(T_1)K} = \overline{\eta(T_1|K)K} = K$. We have $YSXK \subseteq K$ for any $K \in \text{Lat } T_1$, which shows that $YSX \in \text{Alg Lat } T_1 = \text{Alg } T_1$. Hence $YSX = \nu(T_1)^{-1} u(T_1)$ for some $u, v \in H^\infty$, where $v \wedge m_1 = 1$ (cf. [7, Theorem 3.2]). So $\nu(T_1)YSX = u(T_1)$ and we have $\eta(T_2)\nu(T_2)S\eta(T_2) = XY\nu(T_2)SXY = X(\nu(T_1)YSX)Y = X\nu(T_1)Y = u(T_2)XY = u(T_2)\eta(T_2)$. Since as above $\eta(T_2)$ is a quasi-affinity, this implies that $\eta(T_2)\nu(T_2)S = u(T_2)$. Note that $\nu(T_2) \wedge m_2 = 1$. We obtain $S = (\nu(T_2)^{-1} u(T_2) \in \text{Alg } T_2$. This shows that $T_2$ is reflexive, completing the proof.

As a by-product of the preceding proof, we have the following

**Theorem 3.** Let $T_1$ and $T_2$ be $C_0(N)$ contractions on $H_1$ and $H_2$, respectively. If $T_1$ is quasi-similar to $T_2$, then $\text{Lat } T_1 \simeq \text{Lat } T_2$.

**Proof.** Let $X : H_1 \rightarrow H_2$ and $Y : H_2 \rightarrow H_1$ be the intertwining quasi-affinities given in the proof of Theorem 2. For $K_1 \in \text{Lat } T_1$ and $K_2 \in \text{Lat } T_2$ consider the mappings $K_1 \rightarrow \overline{XK_1}$ and $K_2 \rightarrow \overline{YK_2}$. As before we have

\[ \overline{YXK_1} = \overline{\eta(T_1)K_1} = \overline{\eta(T_1|K_1)K_1} = K_1. \]

Similarly, $\overline{XYK_2} = K_2$. We infer that these mappings implement the lattice isomorphisms between $\text{Lat } T_1$ and $\text{Lat } T_2$ and hence $\text{Lat } T_1 \simeq \text{Lat } T_2$.

As a consequence of Theorem 2, to prove Theorem 1 it suffices to consider Jordan operators. The next lemma will be needed in the proof of the necessity part.

**Lemma 4.** Let $T$ be an operator on a Hilbert space $H$. Let $S \in \text{Alg Lat } T \cap \{T\}'$ and $T_1 = T|SH$. Assume that $\text{Alg Lat } T_1 \cap \{T_1\}' = \text{Alg } T_1$. If $T$ is reflexive, so is $T_1$. 

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Proof. Let \( S_1 \in \text{Alg Lat } T_1 \). Consider \( S_1 S \) as an operator on \( H \). For any \( K \in \text{Lat } T \), \( SK \subseteq K \cap \overline{SH} \). Since \( K \cap \overline{SH} \subseteq \text{Lat } T_1 \), we have \( S_1 SK \subseteq S_1(K \cap \overline{SH}) \subseteq K \cap \overline{SH} \subseteq K \). This shows that \( S_1 S \in \text{Alg Lat } T = \text{Alg } T \). Hence \( S_1 TS = S_1 ST = TS_1 S \). It follows that \( S_1 T_1 = T_1 S_1 \) on \( \overline{SH} \), that is, \( S_1 \in \{ T_1 \}' \). We conclude that \( S_1 \in \text{Alg Lat } T_1 \cap \{ T_1 \}' = \text{Alg } T_1 \) and hence \( T_1 \) is reflexive.

To prove the sufficiency part, we essentially follow the same line of arguments as given by Deddens and Fillmore [1] for reflexive linear transformations. The next two lemmas are analogous to part of Theorem 2 and its Corollary in [1], respectively.

**Lemma 5.** Let \( T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k) \) be a Jordan operator defined on \( H = (H^2 \oplus \varphi_1 H^2) \oplus \cdots \oplus (H^2 \oplus \varphi_k H^2) \) and let \( \psi = \varphi_1/\varphi_2 \). If \( S \in \text{Alg Lat } T \), then there exist an outer \( \eta \in H^\infty \) and \( \delta \in H^\infty \) such that \( \eta(T)S = \delta(T) + D \), where \( D \) is an operator on \( H \) satisfying
\[
D[(\xi H^2 \oplus \varphi_1 H^2) \oplus (H^2 \oplus \varphi_2 H^2) \oplus \cdots \oplus (H^2 \oplus \varphi_k H^2)]
\subseteq (\xi \varphi_2 H^2 \oplus \varphi_1 H^2) \oplus 0 \oplus \cdots \oplus 0 \quad \text{for any } \xi \psi.
\]

Proof. Let \( T_j = S(\varphi_j) \), \( H_j = H^2 \oplus \varphi_j H^2 \) and let \( P_j \) denote the (orthogonal) projection from \( H^2 \) onto \( H_j \), \( j = 1, 2, \ldots, k \). For brevity of notation, we identify \( H_j \) as a subspace of \( H \) in the natural way. Let \( e = P_1(1) \in H_1 \) and \( h = Se \in H_1 \), since \( S \) leaves \( H_1 \) invariant. Let
\[
h(\lambda) = h(\lambda)h_\varepsilon(\lambda)
= h_\varepsilon(\lambda) \exp \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} k(t) \, dt \right]
\]
for \( |\lambda| < 1 \),
where \( h_\varepsilon \) and \( h_\varepsilon \) are the inner and outer parts of \( h \), and \( k(t) = \log|h_\varepsilon(t)| \) a.e. Fix \( M > 0 \) and let \( \alpha = \{ t : |h_\varepsilon(t)| > M \} \). Let
\[
\eta(\lambda) = \exp \left[ \frac{1}{2\pi} \int_\alpha \frac{e^{it} + \lambda}{e^{it} - \lambda} (-k(t)) \, dt \right]
\]
for \( |\lambda| < 1 \), and \( \delta = \eta h \). Then it is easily seen that \( \eta, \delta \in H^\infty \) and \( \eta(T)Se = \delta(T)e \). Let \( D = \eta(T)S - \delta(T) \). Then \( De = 0 \).

We first check that \( D(H_2 \oplus \cdots \oplus H_k) = \{0\} \). Let \( f \in H^\infty \) and consider the element \( P_j(f) \) in \( H_j, j = 2, 3, \ldots, k \). Let \( W \) and \( U \) be the invariant subspaces for \( T \) generated by \( P_j(f) \) and \( e \oplus P_j(f) \in H_1 \oplus H_j \), respectively. Let \( g \in W \cap U \subseteq H_j \). Then there exists a sequence of polynomials \( \{ p_n \} \) such that \( p_n(T)(e \oplus P_j(f)) \to 0 \) \( \oplus g \) as \( n \to \infty \). Hence \( P_1(p_n) = p_n(T)e \to 0 \) and \( P_j(p_nf) = p_n(T)P_j(f) \to g \), which imply that \( P_j(p_n) = P_jP_1(p_n) \to 0 \) and \( f(T)P_j(p_n) \to g \). It follows that \( g = 0 \), whence \( W \cap U = \{0\} \). Since \( De = 0 \), we have \( D(P_j(f)) = D(e \oplus P_j(f)) \in W \cap U = \{0\} \). Therefore \( D(P_j(f)) = 0 \). Note that \( \{ P_j(f) : f \in H^\infty \} \) is dense in \( H_j \). We conclude that \( D(H_j) = \{0\} \) for \( j = 2, 3, \ldots, k \). Hence \( D(H_2 \oplus \cdots \oplus H_k) = \{0\} \), as asserted.
Next we show that $D(\xi H^2 \ominus \varphi_1 H^2) \subseteq \xi \varphi_2 H^2 \ominus \varphi_1 H^2$ for any $\xi \varphi$. Let $W_1 = \xi H^2 \ominus \varphi_1 H^2$ and $U_1 = \{ P_1(\xi f) \oplus P_2(f) : f \in H^2 \}$. For $g = \xi f \in W_1$, $Dg = D(P_1(\xi f) \oplus P_2(f)) \in W_1 \cap U_1$. Thus to complete the proof it suffices to show that $W_1 \cap U_1 \subseteq \xi \varphi_2 H^2 \ominus \varphi_1 H^2$. Let $w \in W_1 \cap U_1$. There exists a sequence $\{ f_n \} \subseteq H^2$ such that $P_1(\xi f_n) \oplus P_2(f_n) \to w \oplus 0$ as $n \to \infty$. Assume that $f_n = g_n + \varphi_2 h_n$, where $g_n \in H^2 \ominus \varphi_2 H^2$ and $h_n \in H^2$ for each $n$. We infer that $P_1(\xi g_n + \xi \varphi_2 h_n) \to w$ and $g_n \to 0$. Thus $w - P_1(\xi \varphi_2 h_n) = (w - P_1(\xi g_n + \xi \varphi_2 h_n)) + P_1(\xi g_n) \to 0$. It follows that $w \in \xi \varphi_2 H^2 \ominus \varphi_1 H^2$, completing the proof.

**Lemma 6.** Let $T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k)$ be a Jordan operator defined on $H = (H^2 \ominus \varphi_1 H^2) \oplus \cdots \oplus (H^2 \ominus \varphi_k H^2)$ and let $\varphi = \varphi_1 / \varphi_2$. Then $T$ is reflexive if and only if $S(\varphi)$ is.

**Proof.** **Necessity.** Note that $T(\varphi H^2)H$ is unitarily equivalent to $S(\varphi)$. (An explicit proof can be found in [6, pp. 315–316].) Since $\varphi(\varphi H^2) \ominus \varphi_2 H^2$ and $\varphi_2 \ominus \varphi H^2$, the reflexivity of $T$ implies that of $S(\varphi)$ by Lemma 4.

**Sufficiency.** Let $T$, $H_j$ and $P_j$ be as in the proof of Lemma 5 and let $S \in \text{Alg Lat} T$. By Lemma 5, there exist an outer $\eta \in H^\infty$ and $\delta \in H^\infty$ such that $\eta(T)S = \delta(T) + D$, where $D$ satisfies

$$D[(\xi H^2 \ominus \varphi_1 H^2) \oplus H_2 \oplus \cdots \oplus H_k] \subseteq (\xi \varphi_2 H^2 \ominus \varphi_1 H^2) \oplus H_2 \oplus \cdots \oplus H_k$$

for any $\xi \varphi$. Let $D_1 = D[H^2 \ominus \varphi H^2]$ and $D_2 = D[\varphi H^2 \ominus \varphi_1 H^2] \oplus H_2 \oplus \cdots \oplus H_k$. Since $D(\varphi H^2 \ominus \varphi_1 H^2) \subseteq \xi \varphi_2 H^2 \ominus \varphi_1 H^2 = \{ 0 \}$ and $D(\varphi_2 \ominus \varphi_1 H^2) = \{ 0 \}$, we have $D_2 = 0$. On the other hand, for any $\xi \varphi$ consider the subspace $\xi H^2 \ominus \varphi H^2$ in $\text{Lat} S(\varphi)$. Note that $\xi H^2 \ominus \varphi H^2 \subseteq \xi H^2 \ominus \varphi_1 H^2$. Hence from the property of $D$ we infer that $D_1(\xi H^2 \ominus \varphi H^2) \subseteq \xi \varphi_2 H^2 \ominus \varphi_1 H^2$. Thus the operator $D'$ defined on $H^2 \ominus \varphi H^2$ by $D'f = \varphi_2 D_1 f$ for $f \in H^2 \ominus \varphi H^2$ is in $\text{Alg Lat} S(\varphi)$. By the reflexivity of $S(\varphi)$, there exists $\rho$ in $H^\infty$ such that $D'f = \rho(S(\varphi)f)$ for all $f \in H^2 \ominus \varphi H^2$. It follows that $D_1 f = P_2(\rho f)$, where $P$ denotes the projection from $H^2$ onto $H^2 \ominus \varphi H^2$. For any $h \in H$, $h = f + g$ where $f \in H^2 \ominus \varphi H^2$ and $g = g_1 \oplus \cdots \oplus g_k \in (\varphi H^2 \ominus \varphi_1 H^2) \oplus H_2 \oplus \cdots \oplus H_k$. We deduce that $(\varphi_2 \rho)(T)h = (\varphi_2 \rho)(T) = P_1(\rho)h = P_1(\rho)h$. Consequently, $Dh = D_1 f + D_2 g = (\varphi_2 \rho)(T)h$. This shows that $D = (\varphi_2 \rho)(T)$ and hence $\eta(T)S = \delta(T) + (\varphi_2 \rho)(T)h$. Since $\eta$ is outer, we conclude that $S \in \{ T \}' = \text{Alg T}$ (cf. [7, Theorem 3.2]). Thus $T$ is reflexive, completing the proof.

Now Theorem 1 follows from Theorem 2 and Lemma 6. The condition in Theorem 1 was first formulated by C. Foiaş for general $C_0$ contractions in a private communication to the author. He also proved the necessity part. However our presentation here is more simplified.

**References**


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