ON THE REFLEXIVITY OF C₀(N) CONTRACTIONS

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Abstract. Let T be a C₀(N) contraction on a separable Hilbert space and let
J = S(φ₁) ⊕ S(φ₂) ⊕ ⋯ ⊕ S(φₖ) be its Jordan model, where φ₁, φ₂, ⋯, φₖ are
inner functions satisfying φⱼ|φⱼ₋₁ = ₁ for j = 2, 3, ⋯, k, and S(φⱼ) denotes the com-
pression of the shift on H² ⊗ φⱼH², j = 1, 2, ⋯, k. In this note we show that T is
reflexive if and only if S(φ₁/φ₂) is.

In this note we only consider bounded linear operators defined on complex,
separable Hilbert spaces. For each operator T, let {T}', {T}'' and Alg T denote
the commutant, double commutant and the weakly closed algebra generated by T
and I, respectively. Let Lat T denote the lattice of invariant subspaces of T and
Alg Lat T denote the (weakly closed) algebra of operators which leave all the
subspaces in Lat T invariant. Recall that T is reflexive if and only if Alg Lat T =
Alg T. In [1] Deddens and Fillmore characterized reflexive operators on finite-di-
mensional spaces in terms of their Jordan canonical forms. Now we generalize their
result to C₀(N) contractions. More specifically, we prove the following

Theorem 1. If T is a C₀(N) contraction and J = S(φ₁) ⊕ S(φ₂) ⊕ ⋯ ⊕ S(φₖ) is
its Jordan model, then T is reflexive if and only if S(φ₁/φ₂) is.

A contraction T (||T|| < 1) on a Hilbert space is of class C₀(N) for some integer
N > 1 if there exists an inner function φ such that φ(T) = 0 and the defect indices
of T, dₜ = rank(I - T*T)¹/² and dᵣ = rank(I - TT*)¹/², are both equal to some
M < N. A C₀(N) contraction is unitarily equivalent to the operator T defined on
H = Hᴺ ⊗ ΘₚHᴺ by Tƒ = P(eₙƒ) for ƒ ∈ H, where Hᴺ denotes the standard
Hardy space of Cᴺ-valued functions defined on the unit circle, Θₚ is the character-
stic function of T, and P denotes the (orthogonal) projection from Hᴺ onto H (cf.
[5, Chapter VI]). Two operators T₁, T₂ are quasi-similar if there exist one-to-one
operators X and Y with dense ranges (called quasi-affinities) such that XT₁ = T₂X
and YT₂ = T₁Y. A C₀(N) contraction is quasi-similar to a uniquely determined
Jordan operator (called its Jordan model) J = S(φ₁) ⊕ S(φ₂) ⊕ ⋯ ⊕ S(φₖ),
where φ₁, φ₂, ⋯, φₖ are inner functions satisfying φⱼ|φⱼ₋₁ = ₁ for j = 2, 3, ⋯, k, and
S(φⱼ) denotes the operator defined on H² ⊗ φⱼH² by S(φⱼ)f = Pⱼ(eᵣƒ) for ƒ ∈ H²
⊗ φⱼH², Pⱼ being the (orthogonal) projection from H² onto H² ⊗ φⱼH², j =
1, 2, ⋯, k (cf. [4]). For ξ and η in H∞, ξ \ η = 1 denotes that ξ and η have no
nontrivial common inner divisor.

Received by the editors February 6, 1979 and, in revised form, July 11, 1979.
Key words and phrases. C₀(N) contraction, reflexive operator, Jordan model for C₀(N) contractions,
quasi-similarity.

This research was partially supported by National Science Council of Taiwan, Republic of China.

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We start the proof of Theorem 1 by showing that for $C_0(N)$ contractions, the property of reflexivity is preserved under quasi-similarities. This generalizes Corollary 4.5 in [7].

**Theorem 2.** Let $T_1$ and $T_2$ be $C_0(N)$ contractions on $H_1$ and $H_2$, respectively. Assume that $T_1$ is quasi-similar to $T_2$. Then $T_1$ is reflexive if and only if $T_2$ is.

**Proof.** We may assume that $T_1$ and $T_2$ are defined on $H_1 = H_{\Omega_1} \oplus \Theta_1 H_{\Lambda_1}$ and $H_2 = H_{\Omega_2} \oplus \Theta_2 H_{\Lambda_2}$ by $T_1 f = P_1(e^{\lambda_1} f)$ and $T_2 g = P_2(e^{\lambda_2} g)$, respectively, where $f \in H_1$ and $g \in H_2$. Since $T_1$ and $T_2$ are quasi-similar to each other, there exist bounded analytic functions $\Phi$ and $\Psi$ such that $\Phi \Theta_1 = \Theta_2 \Psi$ and $(\det \Phi)(\det \Psi) \wedge (\det \Theta_1)(\det \Theta_2) = 1$ (cf. [3] and [2]). Let $\Phi'$ denote the algebraic adjoint of $\Phi$. It can be easily verified that the operators $X: H_1 \to H_2$ and $Y: H_2 \to H_1$ defined by $Xf = P_2(\Phi f)$ for $f \in H_1$ and $Yg = P_1((\det \Phi')\Phi'g)$ for $g \in H_2$ implement the quasi-affinities intertwining $T_1$ and $T_2$ (cf. [2, Theorem 2]). Moreover, we have $YX = (\eta(T_1)$ and $XY = (\eta(T_2)$, where $\eta = (\det \Phi)(\det \Psi)$. Let $m_1$ and $m_2$ denote the minimal functions of $T_1$ and $T_2$, respectively. From the quasi-similarity of $T_1$ and $T_2$ we have $m_1 = m_2$.

Assume that $T_1$ is reflexive. Let $S \in \text{Alg Lat } T_2$ and $K \in \text{Lat } T_1$. Then $YSXK \subseteq \overline{Y \eta(T_1)K} = \eta(T_1)K$. $\eta \wedge (\det \Theta_1) = 1$ implies that $\eta \wedge m_1 = 1$ (cf. [5, Theorem VI.5.2]). In particular, $\eta$ and the minimal function of $T_1|K$ have no nontrivial common inner divisor. Thus $\eta(T_1|K)$ is a quasi-affinity (cf. [7, Theorem 2.3]) and therefore $\overline{\eta(T_1|K)K} = \eta(T_1|K)K = K$. We have $YSXK \subseteq K$ for any $K \in \text{Lat } T_1$, which shows that $YSX \in \text{Alg Lat } T_1 = \text{Alg } T_1$. Hence $YSX = v(T_1)^{-1}u(T_1)$ for some $u, v \in H^\infty$, where $v \wedge m_1 = 1$ (cf. [7, Theorem 3.2]). So $v(T_1)YSX = u(T_1)$ and we have $\eta(T_2)v(T_2)S\eta(T_2) = XYv(T_2)SXY = X(v(T_1)YSX)Y = Xv(T_1)Y = u(T_2)XY = u(T_2)$. Since as above $\eta(T_2)$ is a quasi-affinity, this implies that $\eta(T_2)v(T_2)S = u(T_2)$. Note that $\langle \nu \rangle \wedge m_2 = 1$. We obtain $S = (\eta(T_2)^{-1}v(T_2) \in \text{Alg } T_2$. This shows that $T_2$ is reflexive, completing the proof.

As a by-product of the preceding proof, we have the following

**Theorem 3.** Let $T_1$ and $T_2$ be $C_0(N)$ contractions on $H_1$ and $H_2$, respectively. If $T_1$ is quasi-similar to $T_2$, then $\text{Lat } T_1 \cong \text{Lat } T_2$.

**Proof.** Let $X: H_1 \to H_2$ and $Y: H_2 \to H_1$ be the intertwining quasi-affinities given in the proof of Theorem 2. For $K_1 \in \text{Lat } T_1$ and $K_2 \in \text{Lat } T_2$ consider the mappings $K_1 \to \overline{XK_1}$ and $K_2 \to \overline{YK_2}$. As before we have

$$\overline{Y \eta(T_1)K_1} = \eta(T_1)K_1 = \overline{\eta(T_1|K_1)K_1} = K_1.$$ 

Similarly, $\overline{X \eta(T_1)K_2} = K_2$. We infer that these mappings implement the lattice isomorphisms between $\text{Lat } T_1$ and $\text{Lat } T_2$ and hence $\text{Lat } T_1 \cong \text{Lat } T_2$.

As a consequence of Theorem 2, to prove Theorem 1 it suffices to consider Jordan operators. The next lemma will be needed in the proof of the necessity part.

**Lemma 4.** Let $T$ be an operator on a Hilbert space $H$. Let $S \in \text{Alg Lat } T \cap \{T\}'$ and $T_1 = T|S^H$. Assume that $\text{Alg Lat } T_1 \cap \{T_1\}' = \text{Alg } T_1$. If $T$ is reflexive, so is $T_1$. 

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Proof. Let \( S_1 \in \text{Alg Lat } T_1 \). Consider \( S_1S \) as an operator on \( H \). For any \( K \in \text{Lat } T, SK \subseteq K \cap S^2H \). Since \( K \cap S^2H \in \text{Lat } T_1 \), we have \( S_1SK \subseteq S_1(K \cap S^2H) \subseteq K \cap S^2H \subseteq K \). This shows that \( S_1S \in \text{Alg Lat } T = \text{Alg } T \). Hence \( S_1TS = S_1ST = TS_1S \). It follows that \( S_1T_1 = T_1S_1 \) on \( S^2H \), that is, \( S_1 \in \langle T_1 \rangle' \). We conclude that \( S_1 \in \text{Alg Lat } T_1 \cap \langle T_1 \rangle' = \text{Alg } T_1 \) and hence \( T_1 \) is reflexive.

To prove the sufficiency part, we essentially follow the same line of arguments as given by Deddens and Fillmore [1] for reflexive linear transformations. The next two lemmas are analogous to part of Theorem 2 and its Corollary in [1], respectively.

Lemma 5. Let \( T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k) \) be a Jordan operator defined on \( H = (H_2 \oplus \varphi_1H^2) \oplus \cdots \oplus (H_2 \oplus \varphi_kH^2) \) and let \( \psi = \varphi_1/\varphi_2 \). If \( S \in \text{Alg Lat } T \), then there exist an outer \( \eta \in H^\infty \) and \( \delta \in H^\infty \) such that \( \eta(T)S = \delta(T) + D \), where \( D \) is an operator on \( H \) satisfying
\[
D[\{(\xi H^2 \oplus \varphi_1H^2) \oplus (H^2 \oplus \varphi_2H^2) \oplus \cdots \oplus (H^2 \oplus \varphi_kH^2)\}]
\subseteq \{(\xi H^2 \oplus \varphi_1H^2) \oplus 0 \oplus \cdots \oplus 0\} \text{ for any } \xi \psi.
\]

Proof. Let \( T_j = S(\varphi_j), H_j = H^2 \oplus \varphi_jH^2 \) and let \( P_j \) denote the (orthogonal) projection from \( H^2 \) onto \( H_j, j = 1, 2, \ldots, k \). For brevity of notation, we identify \( H_j \) as a subspace of \( H \) in the natural way. Let \( e = P_1(1) \in H_1 \) and \( h = S e \in H_1 \), since \( S \) leaves \( H_1 \) invariant. Let
\[
h(\lambda) = h(\lambda)h_e(\lambda)
= h(\lambda)\exp\left[\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} k(t) \, dt \right]
\text{ for } |\lambda| < 1,
\]
where \( h \) and \( h_e \) are the inner and outer parts of \( h \), and \( k(t) = \log|h_e(t)| \) a.e. Fix \( M > 0 \) and let \( \alpha = \{t: |h_e(t)| > M\} \). Let
\[
\eta(\lambda) = \exp\left[\frac{1}{2\pi} \int_\alpha \frac{e^{it} + \lambda}{e^{it} - \lambda} (-k(t)) \, dt \right]
\text{ for } |\lambda| < 1,
\]
and \( \delta = \eta \). Then it is easily seen that \( \eta, \delta \in H^\infty \) and \( \eta(T)Se = \delta(T)e \). Let \( D = \eta(T)S - \delta(T) \). Then \( De = 0 \).

We first check that \( D(H_2 \oplus \cdots \oplus H_k) = \{0\} \). Let \( f \in H^\infty \) and consider the element \( P_j(f) \) in \( H_j, j = 2, 3, \ldots, k \). Let \( W \) and \( U \) be the invariant subspaces for \( T \) generated by \( P_j(f) \) and \( e \oplus P_j(f) \in H_j \oplus H_j \), respectively. Let \( g \in W \cap U \subset H_j \). Then there exists a sequence of polynomials \( \{p_n\} \) such that \( p_n(T)(e \oplus P_j(f)) \to 0 \) \( \oplus g \) as \( n \to \infty \). Hence \( P_1(p_n) = p_n(T)e \to 0 \) and \( P_1(p_n) = p_n(T)P_j(f) \to g \), which imply that \( P_j(p_n) = P_jP_1(p_n) \to 0 \) and \( f(T)P_j(p_n) \to g \). It follows that \( g = 0 \), whence \( W \cap U = \{0\} \). Since \( De = 0 \), we have \( D(P_j(f)) = D(e \oplus P_j(f)) \in W \cap U = \{0\} \). Therefore \( D(P_j(f)) = 0 \). Note that \( \langle P_j(f): f \in H^\infty \rangle \) is dense in \( H_j \). We conclude that \( D \bar{H}_j = \{0\} \) for \( j = 2, 3, \ldots, k \). Hence \( D(H_2 \oplus \cdots \oplus H_k) = \{0\} \), as asserted.
Next we show that \( D(\xi H^2 \ominus \varphi_1 H^2) \subseteq \xi \varphi_2 H^2 \ominus \varphi_1 H^2 \) for any \( \xi \psi \). Let \( W_1 = \xi H^2 \ominus \varphi_1 H^2 \) and \( U_1 = \{ P_1(\xi f) \oplus P_2(f) : f \in H^2 \} \). For \( g = \xi f \in W_1 \), \( Dg = D(P_1(\xi f) \oplus P_2(f)) \in W_1 \cap U_1 \). Thus to complete the proof it suffices to show that \( W_1 \cap U_1 \subseteq \xi \varphi_2 H^2 \ominus \varphi_1 H^2 \). Let \( w \in W_1 \cap U_1 \). There exists a sequence \( \{ f_n \} \subseteq H^2 \) such that \( P_1(\xi f_n) \oplus P_2(f_n) \to w \oplus 0 \) as \( n \to \infty \). Assume that \( f_n = g_n + \varphi_2 h_n \), where \( g_n \in H^2 \ominus \varphi_2 H^2 \) and \( h_n \in H^2 \) for each \( n \). We infer that \( P_1(\xi g_n + \varphi_2 h_n) \to w \) and \( g_n \to 0 \). Thus \( w - P_1(\xi \varphi_2 h_n) = (w - P_1(\xi g_n + \varphi_2 h_n)) + P_1(\xi g_n) \to 0 \). It follows that \( w \in \xi \varphi_2 H^2 \ominus \varphi_1 H^2 \) completing the proof.

**Lemma 6.** Let \( T = S(\varphi_1) \oplus \cdots \oplus S(\varphi_k) \) be a Jordan operator defined on \( H = (H^2 \ominus \varphi_1 H^2) \oplus \cdots \oplus (H^2 \ominus \varphi_k H^2) \) and let \( \psi = \varphi_1/\varphi_2 \). Then \( T \) is reflexive if and only if \( S(\psi) \) is.

**Proof. Necessity.** Note that \( T \varphi_2(T)H \) is unitarily equivalent to \( S(\psi) \). (An explicit proof can be found in [6, pp. 315–316].) Since \( \varphi_2(T) \in \text{Alg Lat } T \cap \{ T \}' \) and \( \text{Alg Lat } S(\psi) \cap \{ S(\psi) \}' = \text{Alg } S(\psi) \), the reflexivity of \( T \) implies that of \( S(\psi) \) by Lemma 4.

**Sufficiency.** Let \( T_j, H_j \) and \( P_j \) be as in the proof of Lemma 5 and let \( S \in \text{Alg Lat } T \). By Lemma 5, there exist an outer \( \eta \in H^\infty \) and \( \delta \in H^\infty \) such that \( \eta(T)S = \delta(T) + D \), where \( D \) satisfies

\[
D\left[ (\xi H^2 \ominus \varphi_1 H^2) \oplus H_2 \oplus \cdots \oplus H_k \right]
\subseteq (\xi \varphi_2 H^2 \ominus \varphi_1 H^2) \oplus 0 \oplus \cdots \oplus 0 \quad \text{for any } \xi \psi.
\]

Let \( D_1 = D|H^2 \ominus \psi H^2 \) and \( D_2 = D|(\psi H^2 \ominus \varphi_1 H^2) \oplus H_2 \oplus \cdots \oplus H_k \). Since \( D(\psi H^2 \ominus \varphi_1 H^2) \subseteq \xi \varphi_2 H^2 \ominus \varphi_1 H^2 = \{0\} \) and \( D(H_2 \oplus \cdots \oplus H_k) = \{0\} \), we have \( D_2 = 0 \). On the other hand, for any \( \xi \psi \) consider the subspace \( \xi H^2 \ominus \psi H^2 \) in \( \text{Lat } S(\psi) \). Note that \( \xi H^2 \ominus \psi H^2 \subseteq \xi H^2 \ominus \varphi_1 H^2 \). Hence from the property of \( D \) we infer that \( D_1(\xi H^2 \ominus \psi H^2) \subseteq \xi \varphi_2 H^2 \ominus \varphi_1 H^2 \). Thus the operator \( D' \) defined on \( H^2 \ominus \psi H^2 \) by \( D'f = \varphi_2 D_1 f \) for \( f \in H^2 \ominus \psi H^2 \) is in \( \text{Alg Lat } S(\psi) \). By the reflexivity of \( S(\psi) \), there exists \( \rho \in H^\infty \) such that \( D'f = \rho(S(\psi))f \) for all \( f \in H^2 \ominus \psi H^2 \). It follows that \( D_1 f = \varphi_2(\rho(\varphi_1 f)) = P_1(\varphi_2 \varphi_1 f) \), where \( \rho \) denotes the projection from \( H^2 \) onto \( H^2 \ominus \psi H^2 \). For any \( h \in H, h = f + g \) where \( f \in H^2 \ominus \psi H^2 \) and \( g = g_1 \oplus \cdots \oplus g_k \in (\psi H^2 \ominus \varphi_1 H^2) \oplus H_2 \oplus \cdots \oplus H_k \). We deduce that \( (\varphi_2 \rho)(T)h = (\varphi_2 \rho)(T_1)(f + g_1) = P_1(\varphi_2 \rho f + \varphi_2 \rho g_1) = P_1(\varphi_2 \rho f) = D_1 f \). Consequently, \( Dh = D_1 f + D_2 g = (\varphi_2 \rho)(T)h \). This shows that \( D = (\varphi_2 \rho)(T) \) and hence \( \eta(T)S = \delta(T) + (\delta + \varphi_2 \rho)(T) \). Since \( \eta \) is outer, we conclude that \( S \in \{ T \}' = \text{Alg } T \) (cf. [7, Theorem 3.2]). Thus \( T \) is reflexive, completing the proof.

Now Theorem 1 follows from Theorem 2 and Lemma 6. The condition in Theorem 1 was first formulated by C. Foiaş for general \( C_0 \) contractions in a private communication to the author. He also proved the necessity part. However our presentation here is more simplified.

**References**


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