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BEST L^p-APPROXIMATION OF GENERALIZED BIAXISYMMETRIC POTENTIALS

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ABSTRACT. Let F be a real-valued generalized biaxisymmetric potential (GBASP) in L^p $(p > 1)$ on Σ, the open unit sphere about the origin. Convergence of a sequence of best harmonic polynomial approximates to F in L^p identifies those F that harmonically continue as entire function GBASP and determines their order and type. The analysis utilizes the Bergman and Gilbert Integral Operator Method to extend results from classical function theory on the best polynomial approximation of analytic functions of one complex variable.

Introduction. Let $F = F(x, y)$ be a real-valued regular solution to the generalized biaxially symmetric potential equation,

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + (2\alpha + 1)/x\partial/\partial x + (2\beta + 1)/y\partial/\partial y)F = 0, \qquad \alpha > \beta > -1/2, \quad (1)$$

(α, β) fixed in a neighborhood of the origin where the analytic Cauchy data, $F_x(0, y) = F_y(x, 0) = 0$, is satisfied along the singular lines. Such functions with even harmonic extensions are referred to as generalized biaxisymmetric potentials (GBASP), having local expansions of the form

$$F(x, y) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(x, y),$$

$$R_n^{(\alpha,\beta)}(x, y) = (x^2 + y^2)^n P_n^{(\alpha,\beta)}((x^2 - y^2)/(x^2 + y^2))/P_n^{(\alpha,\beta)}(1) \qquad (2)$$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials [1], [17]. Suitable limits of the parameters (α, β), after quadratic transformations [1] as necessary, produce various special functions from the $R_n^{(\alpha,\beta)}$. For example, $\alpha = \beta = 0$ gives the zonal harmonics so that F interprets as an axisymmetric potential on E^3 and $\alpha = \beta = -1/2$ gives the even circular harmonics on E^2 where the interpretation is $F = \operatorname{Re} f$, f is real analytic. Further examples can be found in [1]. The Euler-Poisson-Darboux equation, arising in gas dynamics, is viewed in terms of equation (1) after a transformation [7, p. 223]. The F then suggest generalizations of analytic functions and have a variety of physical interpretations [4], [5], [7], [9]; also refer to the Method of Ascent [6]. Thus, of special interest are global properties characterizing solutions to this partial differential equation that are determined from local properties.

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This paper identifies those GBASP $F \in L^p(\Sigma)$, Σ the open unit sphere about the origin, that harmonically continue as entire function GBASP. The characteristic feature follows from the rate of convergence of a sequence of best GBASP polynomial approximates to F in $L^p(\Sigma)$. For entire function GBASP, the growth of this sequence is used to calculate the order and type which are defined as in function theory. Basic to this analysis is application of the Bergman [2] and Gilbert [4], [5], [6] Integral Operator Method to extend corresponding classifications of associated analytic functions due to A. R. Reddy [14] and I. I. Ibragimov and N. I. Sihaliev [8] which are extensions of S. N. Bernstein's classical theorem [3]. For information on the location of singularities and growth of GBASP regular in Σ and continuous on $\mathrm{cl}(\Sigma)$, see [11], [12], [13] for approximates in the uniform norm taken relative to harmonic polynomials or Newtonian potentials as in the Bernstein theorem and its extensions [15], [16], [18].

Preliminaries. Fundamental to this study are the invertible integral operators $\mathcal{K}_{\alpha,\beta}$ and $\mathcal{K}_{\alpha,\beta}^{-1}$ developed in [12] that locally associate regular GBASP F, equation (2), and the unique analytic function f,

$$f(z) = \sum_{n=0}^{\infty} a_n z^{2n} \tag{3}$$

as follows

(i) $F = \mathcal{K}_{\alpha,\beta}(f)$ where

$$F(x, y) = \int_0^1 \int_0^{\pi} f(\zeta) k_{\alpha,\beta}(t, s)\, ds\, dt,$$

$$k_{\alpha,\beta}(t, s) = \gamma_{\alpha,\beta}(1 - t^2)^{\alpha-\beta-1} t^{2\beta+1} (\sin s)^{2\alpha},$$

$$\zeta^2 = x^2 - y^2 t^2 + i2xyt \cos s;$$

(ii) $f = \mathcal{K}_{\alpha,\beta}^{-1}(F)$ where

$$f(z) = \int_{-1}^{+1} F\left(r\xi, r\sqrt{1 - \xi^2}\right) S_{\alpha,\beta}(z/r, \xi)\, d\xi,$$

$$S_{\alpha,\beta}(\tau, \xi) = \eta_{\alpha,\beta} \frac{(1 - \tau)}{(1 + \tau)^{\alpha+\beta+2}}\, {}_2F_1\left(\frac{\alpha + \beta + 2}{2}; \frac{\alpha + \beta + 3}{2}; \beta + 1; \frac{2\tau(1 + \xi)}{(1 + \tau)^2}\right).$$

The normalizations $\mathcal{K}_{\alpha,\beta}(1) = \mathcal{K}_{\alpha,\beta}^{-1}(1) = 1$ are taken. The kernel $S_{\alpha,\beta}(\tau, \xi)$ is analytic on $|\tau| < 1$ for $-1 < \xi < +1$. The local function elements F and f are harmonically/analytically continued by contour deformation by the Envelope Method [5].

REMARK 1. This method [12] verifies that F is regular in $x^2 + y^2 < \sigma^2$ if and only if the associate f is analytic in $x^2 + y^2 < \sigma^2$. The order and type of an entire function GBASP are defined from the maximum modulus $M_r(F) = \sup \{|F(x, y)|: x^2 + y^2 < r^2\}$ as in function theory [10], respectively,

$$\rho(f) = \limsup_{r\to\infty} \frac{\log \log M_r(F)}{\log r}, \qquad \tau(F) = \limsup_{r\to\infty} \frac{\log M_r(F)}{r^{\rho(F)}}.$$

REMARK 2. The orders and types of the GBASP and the associate are respectively equal [12]. Henceforth, the notation $\rho = \rho(F) = \rho(f)$ and $\tau = \tau(F) = \tau(f)$.

REMARK 3. From these facts, the usual coefficient formulas [10] for order and type are inferred [12].

The L^p-characterizations. Let $A_p(\Sigma_\sigma)$ and $a_p(\Sigma_\sigma)$ denote the spaces of GBASP and associates that remain regular and respectively analytic on Σ_σ: $x^2 + y^2 < \sigma^2$ with finite norms

$$\|F_\sigma\|_\sigma = \left(\iint_{x^2+y^2<\sigma^2} |F_\sigma|^p \, dx \, dy \right)^{1/p},$$

$$\|f_\sigma\|_\sigma = \left(\iint_{x^2+y^2<\sigma^2} |f_\sigma|^p \, dx \, dy \right)^{1/p}$$

for fixed $p > 1$. For each integer n, define the sets of polynomials

$$\mathfrak{p}_n = \left\{ \sum_{k=0}^{n} a_k z^{2k} : a_k - \text{real} \right\}, \qquad \mathcal{P}_n = \{ \mathcal{K}_{\alpha,\beta}(p_n) : p_n \in \mathfrak{p}_n \}$$

with $\mathfrak{p}_{\sigma,n} = \{ p_{\sigma,n}(z) = p_n(z/\sigma) : p_n \in \mathfrak{p}_n \}$ and $\mathcal{P}_{\sigma,n} = \{ \mathcal{K}_{\alpha,\beta}(p_{\sigma,n}) : p_{\sigma,n} \in \mathfrak{p}_{\sigma,n} \}$. Finally, define the best polynomial approximates to the GBASP and associate as

$$\delta_n^{(p)}(f_\sigma) = \inf \{ \|f_\sigma - p_{\sigma,n}\|_\sigma : p_{\sigma,n} \in \mathfrak{p}_{\sigma,n} \},$$

$$\Delta_n^{(p)}(F_\sigma) = \inf \{ \|F_\sigma - P_{\sigma,n}\|_\sigma : P_{\sigma,n} \in \mathcal{P}_{\sigma,n} \}.$$

For each n there is an extremal polynomial $p^*_{\sigma,n} \in \mathfrak{p}_{\sigma,n}$ for which $\delta_n^{(p)}(f_\sigma) = \|f_\sigma = p^*_{\sigma,n}\|_\sigma$ [8]. In the following it will be evident that for each n, $\Delta_n^{(p)}(F_\sigma) = \|F_\sigma - P^*_{\sigma,n}\|_\sigma$, $P^*_{\sigma,n} \in \mathcal{P}_{\sigma,n}$. When $\sigma = 1$ in the above notations, the subscripts σ are dropped. The L^p classifications of GBASP which extend the classical result of I. Ibragimov and N. Sihaliev follow.

THEOREM 1. *Let the GBASP $F \in A_p(\Sigma)$, $p > 1$. Then F harmonically continues as an entire function GBASP if and only if*

$$\lim_{n\to\infty} \left[\Delta_n^{(p)}(F) \right]^{1/n} = 0. \tag{4}$$

PROOF. Let $F \in A_p(\Sigma)$; then for $0 < \sigma < 1$, $F_\sigma \in A_p(\Sigma_\sigma)$. Setting $f_\sigma = \mathcal{K}^{-1}_{\alpha,\beta}(F_\sigma)$ and applying Hölder's inequality to the following estimate, $|f_\sigma| < \int_{-1}^{+1} |F_\sigma| |S_{\alpha,\beta}| \, d\xi$, gives

$$|f_\sigma|^p < \omega^p_{\alpha,\beta} \int_{-1}^{+1} |F_\sigma|^p \, d\xi$$

on $x^2 + y^2 < \sigma^2$ where $\omega_{\alpha,\beta} = \sup \{ |S_{\alpha,\beta}(\tau, \xi)| : |\tau| < \sigma, |\xi| < 1 \}$. Now by Fubini's theorem,

$$\|f_\sigma\|^p < \omega^p_{\alpha,\beta} \int_{-1}^{+1} \left[\iint_{x^2+y^2<\sigma^2} |F_\sigma|^p \, dx \, dy \right] d\xi < 2\omega^p_{\alpha,\beta} \|F_\sigma\|^p_\sigma.$$

Then

$$\|f_\sigma\|_\sigma < C \|F_\sigma\|_\sigma, \qquad C = 2^{1/p} \omega_{\alpha,\beta}. \tag{5}$$

So that, $f_\sigma \in a_p(\Sigma_\sigma)$. Thus the resulting bound

$$\delta_n^{(p)}(f_\sigma) < C\Delta_n^{(p)}(F_\sigma) \tag{6}$$

gives the relation

$$\lim_{n\to\infty} \left[\delta_n^{(p)}(f_\sigma)\right]^{1/n} < \lim_{n\to\infty} \left[\Delta_n^{(p)}(F_\sigma)\right]^{1/n}. \tag{7}$$

When equation (4) is satisfied, we find from equation (7) and [8] that f_σ is entire. Thus f is entire and, by Remark 1, F harmonically continues as an entire function.

To establish the converse, let $F \in A_p(\Sigma)$ be an entire function. Hence, the f_σ is entire as is $F_\sigma = \mathcal{K}_{\alpha,\beta}(f_\sigma)$. Hölder's inequality is applied to the bound $|F_\sigma| < \int_0^1\int_0^\pi |f_\sigma||k_{\alpha,\beta}|\,ds\,dt$ to give

$$|F_\sigma|^p < \gamma_{\alpha,\beta}^p \int_0^1 \int_0^\pi |f_\sigma|^p\,ds\,dt.$$

Applying Fubini's theorem along with a simple estimate gives

$$\begin{aligned} \|F_\sigma\|_\sigma^p &< \gamma_{\alpha,\beta}^p \int_0^1 \int_0^\pi \left[\iint_{x^2+y^2<\sigma^2} |f_\sigma|^p\,dx\,dy \right] dt\,ds \\ &< \gamma_{\alpha,\beta}^p \int_0^1 \int_0^\pi \|f_\sigma\|^p\,dt\,ds, \end{aligned}$$

which is a bound $\|F_\sigma\|_\sigma < K\|f_\sigma\|$, $K = \pi^{1/p}\gamma_{\alpha,\beta}$ leading directly to

$$\Delta_n^{(p)}(F_\sigma) < K\delta_n^{(p)}(f_\sigma). \tag{8}$$

However, because f_σ is entire, the larger term in $\lim_{n\to\infty}[\Delta_n^{(p)}(F_\sigma)]^{1/n} < \lim_{n\to\infty}[\delta_n^{(p)}(f_\sigma)]^{1/n}$ vanishes implying equation (4) and completing the proof.

Next, consider the calculation of the (finite) order. This is given in

THEOREM 2. *Let the $F \in A_p(\Sigma)$, $p > 1$, be an entire function GBASP. The order of F is ρ if and only if*

$$\limsup_{n\to\infty} \frac{n\log n}{-\log\left[\Delta_n^{(p)}(F)\right]} = \rho. \tag{9}$$

PROOF. Let F be as described. Then $F_\sigma \in A_p(\Sigma_\sigma)$ and is entire because of the coefficient formula [12] for order,

$$\rho(F) = \limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|} = \limsup_{n\to\infty} \frac{n\log n}{-\log|a_n/\sigma^n|} = \rho(F_\sigma).$$

Because $f_\sigma = \mathcal{K}_{\alpha,\beta}^{-1}(F_\sigma)$, let $\varepsilon > 0$ be given, and apply Remark 2 to find that

$$\rho - \varepsilon < \frac{n\log n}{-\log\left[\delta_n^{(p)}(f_\sigma)\right]} < \rho + \varepsilon,$$

where the lower estimate holds for infinitely many indices and the upper estimate holds for all but a finite number. After a little manipulation, working with the upper estimate and equation (5) gives $n^n < C^{1/\rho+\varepsilon}[\Delta_n^{(p)}(F_\sigma)]^{1/\rho+\varepsilon}$ for infinitely many indices leading to

$$\limsup_{n\to\infty} \frac{n\log n}{-\log\left[\Delta_n^{(p)}(F_\sigma)\right]} < \rho + \varepsilon. \tag{10}$$

The lower estimate and equation (8) give $K^{-1/\rho-\varepsilon}[\Delta_n^{(p)}(F_\sigma)]^{1/\rho-\varepsilon} < n^n$ for infinitely many indices leading to

$$\rho - \varepsilon < \limsup_{n\to\infty} \frac{n\log n}{-\log\left[\Delta_n^{(p)}(F_\sigma)\right]} \tag{11}$$

which completes the proof when combined with equation (10) since $\Delta_n^{(p)}(F_\sigma) = \Delta_n^{(p)}(F)$.

To verify the converse, assume that equation (9) is satisfied. Then for $\varepsilon > 0$,

$$\rho - \varepsilon < \frac{n\log n}{-\log\left[\Delta_n^{(p)}(F_\sigma)\right]} < \rho + \varepsilon \tag{12}$$

with the larger bound holding for all but finitely many indices and the lower for infinitely many. Combination of the larger and smaller bounds in equation (12) with equations (8) and (6) gives the following estimates with indices interpreted as above. The relations are $[\delta_n^{(p)}(f_\sigma)]^{1/\rho+\varepsilon} < C^{1/\rho+\varepsilon}[\Delta_n^{(p)}(F_\sigma)]^{1/\rho+\varepsilon}$ so that

$$\limsup_{n\to\infty} \frac{n\log n}{-\log\left[\delta_n^{(p)}(f_\sigma)\right]} < \limsup_{n\to\infty} \frac{n\log n}{-\log\left[\Delta_n^{(p)}(f_\sigma)\right]} \tag{13}$$

and also

$$\left[K^{-1}\Delta_n^{(p)}(F_\sigma)\right]^{1/\rho-\varepsilon} < \left[\delta_n^{(p)}(f_\sigma)\right]^{1/\rho-\varepsilon}$$

giving

$$\limsup_{n\to\infty} \frac{n\log n}{-\log\left[\Delta_n^{(p)}(F_\sigma)\right]} < \limsup_{n\to\infty} \frac{n\log n}{-\log\left[\delta_n^{(p)}(f_\sigma)\right]}. \tag{14}$$

The bounds in equations (13) and (14) show that the expression

$$\rho - \varepsilon < \limsup_{n\to\infty} \frac{n\log n}{-\log\left[\delta_n^{(p)}(f_\sigma)\right]} < \rho + \varepsilon,$$

defines the order of f_σ (see [8]) and thus the order of F_σ by Remark 2, establishing equation (9) as the order of F.

Having determined the order, we calculate the type of F. This proceeds along the same lines as that of the order after observing that $\sigma^p\tau(F_\sigma) = \tau(F)$, a relation found from the coefficient formula. The resulting formula is summarized in

THEOREM 3. *Let the $F \in A_p(\Sigma)$, $p > 1$, be an entire function GBASP of order ρ. Then the type of F is τ if and only if*

$$\limsup_{n\to\infty} n\left[\Delta_n^{(p)}(F)\right]^{\rho/n} = \tau e\rho.$$

Remarks on generalizations. The Method of Ascent constructs maps between the linear space of harmonic functions and linear spaces of solutions to more general elliptic partial differential equations via integral transforms whose kernels depend solely on the coefficients of the equation. These transforms suggest upper bounds for order and type and sufficient conditions for the existence of entire function solutions via L^p approximates similar to those used for the GBASP.

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