BALAYAGE DEFINED BY THE NONNEGATIVE CONVEX FUNCTIONS

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ABSTRACT. We study the Choquet order induced on measures on a linear space by the cone of nonnegative convex functions. We are concerned mainly with discrete measures, and the following result is typical. Let \( x_1, \ldots, x_r, y_1, \ldots, y_n \), where \( r < n \), be points in \( \mathbb{R}^d \). Then

\[
\sum_{k=1}^{r} f(x_k) < \sum_{k=1}^{n} f(y_k)
\]

for all nonnegative, continuous, convex functions \( f \) if, and only if, there exists a doubly stochastic matrix \( M \) such that

\[
x_j = \sum_{k=1}^{n} m_{jk} y_k \quad (j = 1, \ldots, r).
\]

In the case \( d = 1 \), this result may be found in the work of L. Mirsky; our methods allow us to place such results in a general setting.

1. The key step. Here we deal with Baire measures (always nonnegative and finite) on a compact subset \( K \) of a Hausdorff space \( E \). We shall write \( \mu(f) \) for the integral of a continuous (real-valued) function \( f \) on \( K \) with respect to the measure \( \mu \), and \( \delta_x \) for the unit mass at \( x \in K \). When \( K \) has a convex structure, we write \( C \) for the cone of continuous convex functions on \( K \) and \( C^+ \) for the nonnegative members of \( C \).

If \( \mu(f) < \nu(f) \) for all \( f \in C^+ \), we follow a common terminology (see, e.g., P. A. Meyer [3, Chapter XI, §3]) in saying that the measure \( \nu \) is a "balayage" of \( \mu \) relative to the class \( C^+ \). The following theorem relates balayage relative to \( C^+ \) to balayage relative to \( C \).

**Theorem 1.** Let \( \mu, \nu \) be two Baire measures on a compact convex subset \( K \) of a locally convex topological vector space \( E \). If \( \nu \) is a balayage of \( \mu \) relative to \( C^+ \), then there exists a Baire measure \( \lambda \) such that \( \nu \) is a balayage of \( \mu + \lambda \) relative to \( C \). Furthermore we may choose \( \lambda \) to be concentrated at a point \( x_0 \in K \) if we wish: \( \lambda = (\nu(1) - \mu(1))\delta_{x_0} \).

**Proof.** By the Riesz representation theorem, each (nonnegative) Baire measure \( \lambda \) corresponds to a continuous linear functional \( \phi \) on \( C(K) \) such that \( \phi(f) > 0 \) for all \( f \) in the (convex) cone \( P \) of positive functions in \( C(K) \). If we find such a \( \phi \) with the additional properties

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\[ \phi(f) < (\nu - \mu)(f) \quad (f \in C^+) \]

and
\[ \phi(1) = (\nu - \mu)(1), \]

then \( \nu \) is a balayage of \( \mu + \lambda \) relative to \( C \), since \( C = \{ f + r1 : f \in C^+, r \in \mathbb{R} \} \).

If \( \mu(1) = \nu(1) \) we can simply put \( \phi = 0 \). Otherwise, setting \( \alpha = \nu - \mu \), we have \( \alpha(1) \geq 0 \) and we can define the linear operator \( T \) on \( C(K) \) by
\[ Tf = f - (\alpha(f)/\alpha(1))1. \]

Now \( f \in C^+ \) implies \( Tf \in C \) and, we claim, \( Tf \notin P \); otherwise we would have
\[ 0 < m = \min Tf, \quad \text{and} \quad Tf - m1 \in C^+ \] so that
\[ 0 < \alpha(Tf - m1) = \alpha(f - (\alpha(f)/\alpha(1))1) - \alpha(m1) = -\alpha(1) < 0, \]
a contradiction. Thus the cone \( TC^+ \) and the open cone \( P \) do not intersect, so that by a well-known form of the separation theorem (see, e.g., W. Rudin [6, Theorem 3.4]) there is a continuous linear functional \( \psi \equiv 0 \) on \( C(K) \) such that \( \psi(f) < \psi(g) \) whenever \( f \in TC^+ \) and \( g \in P \).

Since \( P \) is an open cone and \( \psi \equiv 0 \), it is clear that \( \psi(P) \), being bounded below, must be \((0, \infty)\). If we set \( \phi = (\alpha(1)/\psi(1))\psi \), then \( \phi(1) = \alpha(1) \), and it remains to show that \( \phi(f) < \alpha(f) \) whenever \( f \in C^+ \). But in this case we have ensured that \( \phi(Tf) < 0 \), so that \( \phi(f) < \phi((\alpha(f)/\alpha(1))1) = \alpha(f) \).

Finally we note that we can, if we wish, replace \( \lambda \) by its resultant \( \lambda(1)\delta_{x_0} \); more precisely, let \( x_0 \) be the barycentre of the probability measure \( \lambda/\lambda(1) \) (see R. R. Phelps [5, Proposition 1.1]). By definition, \( x_0 \) is that point in \( K \) such that \( \delta_{x_0}(f) = \lambda(f)/\lambda(1) \) for all continuous affine functions \( f \) on \( K \), and it is well known that the inequality \( \lambda(1)/f(x_0) \leq \lambda(f) \) follows for all \( f \in C \). Thus we may replace \( \lambda \) by \( \lambda(1)\delta_{x_0} = \nu(1) - \mu(1)\delta_{x_0} \). Q.E.D.

2. Application. As we shall make clear in the remarks below, the following theorem provides a common extension of some basic results on balayage for discrete measures.

**Theorem 2.** Let \( x = (x_1, \ldots, x_r) \), \( y = (y_1, \ldots, y_n) \) where \( r < n \) and \( x_k, y_k \) are elements of \( \mathbb{R}^d \). Then the following are equivalent.

(i) \( \Sigma f(x_k) < \Sigma f(y_k) \) for every convex continuous function \( f: K \to \mathbb{R}^+ \), where \( K \) is the convex hull in \( \mathbb{R}^d \) of the \( x \)'s and \( y \)'s.

(ii) \( x = [My] \), for some doubly stochastic matrix \( M \) (here \( [z] \), denotes the vector formed by the first \( r \) components of the vector \( z \) and the product \( My \) is interpreted formally with \( y \) as a column vector).

Remarks. (a) In the one-dimensional case \( (d = 1) \) this result comes from Ch. Davis and L. Mirsky (see [4]). In that case, the well-known relationship between doubly stochastic matrices and the Hardy-Littlewood-Pólya order allows the addition of a third equivalent statement:

(iii) \( x = [z] \), for some \( z \in \mathbb{R}^n \) such that
\[ \sum_{k=1}^{l} z_k^* \leq \sum_{k=1}^{l} y_k^* \]
for \( i = 1, \ldots, n \) with equality holding for \( i = n \), where \( z^* \) and \( y^* \) denote the nonincreasing rearrangements of \( z \) and \( y \).

(b) The general \( d \)-dimensional form of Theorem 2 is due to S. Sherman [7] and C. Stein (see D. Blackwell [1]) in the special case \( r = n \). Note that in this case there is no need to require nonnegative functions in (i) since the inequality is unchanged upon adding any constant to \( f \). We shall prove Theorem 2 by deriving it from the special case of Sherman and Stein.

Proof. (i) \( \Rightarrow \) (ii). In the terminology of Theorem 1, the hypothesis says that 
\[ \nu = \sum_1^n \delta_{x_k} \]
for a balayage of \( \mu = \sum_1^n \delta_{x_k} \) relative to \( C^+ \). By Theorem 1, there exists \( x_0 \) such that \( \nu \) is a balayage of \( \mu + (n-r)\delta_{x_0} \) relative to \( C \). Thus \( z = (x_1, \ldots, x_r, x_0, \ldots, x_0) \) and \( y \) satisfy the hypothesis (i) of the Sherman-Stein theorem, so that \( z = My \) for some doubly stochastic \( M \). Hence \( x = [z] = [My] \).

(ii) \( \Rightarrow \) (i). By the theorem of Sherman and Stein and the nonnegativity of \( f \), we have, with \( z = My \),
\[
\sum_1^r f(y_k) > \sum_1^r f(z_k) > \sum_1^n f(z_k) = \sum_1^n f(x_k).
\]
Q.E.D.

3. A variant. In [2] an analogue of the Sherman-Stein theorem is established for substochastic matrices. One form of this result may be stated as follows (cf. [2, Théorème 8]).

**Theorem 3.** Let \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) where \( x_k \) and \( y_k \) are elements of \( \mathbb{R}^d \). Then the following statements are equivalent.

(i) \[
\sum_{k=1}^n f(x_k) < \sum_{k=1}^n f(y_k)
\]
for every continuous convex function \( f: \mathbb{R}^d \to \mathbb{R} \) such that \( f > f(0) \).

(ii) there exists an \( n \times n \) doubly substochastic matrix \( M \) such that \( x = My \).

The following theorem extends Theorem 3 in the same way that Theorem 2 extends the Sherman-Stein theorem.

**Theorem 4.** Let \( x = (x_1, \ldots, x_r) \), \( y = (y_1, \ldots, y_n) \), where \( r < n \) and the \( x_k \) and \( y_k \) are elements of \( \mathbb{R}^d \). Then the following statements are equivalent.

(i) \[
\sum_{k=1}^r f(x_k) < \sum_{k=1}^n f(y_k)
\]
for every continuous convex function \( f: \mathbb{R}^d \to \mathbb{R}^+ \) such that \( f > f(0) \).

(ii) there exists an \( n \times n \) doubly substochastic matrix \( M \) such that \( x = [My]_r \).

Proof. (ii) \( \Rightarrow \) (i). This follows by an obvious modification of our proof of the corresponding implication in Theorem 2.

(i) \( \Rightarrow \) (ii). For any convex continuous \( f: \mathbb{R}^d \to \mathbb{R} \) such that \( f > f(0) \) the function \( f - f(0) \) satisfies the hypotheses and the resulting inequality clearly implies that
\[
\sum_1^r f(x_1) + (n-r)f(0) < \sum_1^n f(y_k).
\]
Using the implication (i) \( \Rightarrow \) (ii) of Theorem 3, we see that there exists a doubly substochastic matrix \( M \) so that \( (x_1, \ldots, x_r, 0, \ldots, 0) = M(y_1, \ldots, y_n) \). Q.E.D.
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