A SHORT PROOF FOR A.E. CONVERGENCE OF GENERALIZED CONDITIONAL EXPECTATIONS

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ABSTRACT. Let $L_s(\mu)$ be the space of real valued random variables with $\mu(|f|^s) < \infty$, $1 < s < \infty$. Let $C \subset L_s(\mu)$ be a closed convex set. For each $f \in L_s(\mu)$ there exists a unique element $\mu_s(f|C)$ with $\|f - \mu_s(f|C)\|_s < \|f - c\|_s$ for every $c \in C$. Let $C_n$ be a decreasing or increasing sequence of closed convex lattices converging to the closed convex lattice $C_\infty$. We show that $\mu_s(f|C_n) \to \mu_s(f|C_\infty)$ a.e. for every $f \in L_s(\mu)$.

This result contains the results of a.e. convergence of prediction sequences of Ando-Amemiya and the result of Brunk and Johansen of a.e. convergence of conditional expectations given $\sigma$-lattices.

Let $\mu$ be a measure defined on a $\sigma$-algebra $\mathcal{F}$ over $\Omega$. For each $s$ with $1 < s < \infty$ denote by $L_s(\mu)$ the space of equivalence classes of real valued random variables $f$ with $\mu(|f|^s) < \infty$. Then $L_s(\mu)$ is a uniformly convex Banach space. Hence for each closed convex set $C$ and each $f \in L_s(\mu)$ there exists a unique element $\mu_s(f|C)$ fulfilling

(i) $\mu_s(f|C) \in C$,
(ii) $\|f - \mu_s(f|C)\|_s < \|f - c\|_s$ for all $c \in C$,

where $\|f\|_s = [\mu(|f|^s)]^{1/s}$.

If $\mu$ is a probability measure $P$, $s = 2$, $\mathcal{B} \subset \mathcal{F}$ is a sub-$\sigma$-field and $C$ is the system of all square integrable equivalence classes of functions which contain a $\mathcal{B}$-measurable function, then $P_s(f|C)$ is the usual conditional expectation of $f$ given $\mathcal{B}$, i.e. $P_s(f|C) = P^\mathcal{B}f$.

If $\mu$ is a probability measure, $s \neq 2$, $\mathcal{B} \subset \mathcal{F}$ is a sub-$\sigma$-field and $C$ is the system of all equivalence classes of functions which contain a $\mathcal{B}$-measurable function, then $P_s(f|C)$ is the $s$-prediction $P^\mathcal{B}_sf$ of Ando-Amemiya [1].

If $s = 2$, $\mathcal{C} \subset \mathcal{F}$ is a sub-$\sigma$-lattice and $C$ is the system of all square integrable equivalence classes of functions which contain a $\mathcal{C}$-measurable function, then $\mu_2(f|C)$ is the conditional expectation $\mu(f|\mathcal{C})$ of $f$ given $\mathcal{C}$ in the sense of [2].

Let $C_n$ be a decreasing or increasing sequence of closed convex subsets of $L_s(\mu)$. Put $C_\infty = \cap_{n \in \mathbb{N}} C_n$ for the decreasing case and let $C_\infty$ be the closure of $\cup_{n \in \mathbb{N}} C_n$ with respect to $\|\cdot\|_s$ for the increasing case.

Using the definition (i), (ii) of $\mu_s(f|C)$ one directly obtains

$\|f - \mu_s(f|C_n)\|_s \to \|f - \mu_s(f|C_\infty)\|_s$. \hfill (*)

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The following theorem gives a very short proof for the $\mu$-a.e. convergence of $\mu_s(f|C_n)$ to $\mu_s(f|C_\infty)$. The proof is even for the classical conditional expectation $P^Sf$ shorter and more transparent than the known ones. It contains the a.e. convergence result given in [1] and [2].

$C$ is a lattice if $f, g \in C$ imply $f \vee g, f \wedge g \in C$ where $f \vee g [f \wedge g]$ is the pointwise maximum [minimum] of $f$ and $g$.

If $C_n$ is a decreasing or increasing sequence of closed convex lattices then $C_\infty$ is a closed convex lattice too.

**Theorem.** Let $1 < s < \infty$ and $C_n \subset L_s(\mu)$, $n \in \mathbb{N}$, be an increasing or decreasing sequence of closed convex lattices converging to the closed convex lattice $C_\infty$. Then

$$\mu_s(f|C_n) \to \mu_s(f|C_\infty) \text{ } \mu\text{-a.e. for every } f \in L_s(\mu).$$

**Proof.** Let $f, g, h \in L_s(\mu)$. Using the trivial identity

$$|f(g) - g(h) + h(f)| \leq |f(g) - g(h)| + |f(g) - h(f)|$$

we obtain

$$\int |f - g| \wedge |h| \, d\mu + \int |f - g| \wedge |h| \, d\mu = \int |f - g| \wedge |h| \, d\mu + \int |f - h| \wedge |g| \, d\mu. \quad (1)$$

Let $C_n$ be increasing. We shall show that for $n < m$

$$\|f - \mu_s(f|C_n) \wedge \ldots \wedge \mu_s(f|C_m)\|_s < \|f - \mu_s(f|C_n) \wedge \ldots \wedge \mu_s(f|C_m)\|_s. \quad (2)$$

We apply (1) to $g = \mu_s(f|C_n) \wedge \ldots \wedge \mu_s(f|C_m)$, $h = \mu_s(f|C_m)$. If (2) would be false, (1) implies

$$\|f - g \vee h\|_s = \|f - [\mu_s(f|C_n) \wedge \ldots \wedge \mu_s(f|C_m)] \vee \mu_s(f|C_m)\|_s \leq \|f - \mu_s(f|C_n) \wedge \ldots \wedge \mu_s(f|C_m)\|_s,$$

which contradicts

$$[\mu_s(f|C_n) \wedge \ldots \wedge \mu_s(f|C_m)] \vee \mu_s(f|C_m) \in C_m.$$

From (2) we obtain for all $n < m$

$$\|f - \mu_s(f|C_n) \wedge \ldots \wedge \mu_s(f|C_m)\|_s < \|f - \mu_s(f|C_n)\|_s. \quad (3)$$

Using the lemma of Fatou ($m \to \infty$), (3) implies that for all $n$

$$\left\| f - \wedge_{m \geq n} \mu_s(f|C_n) \right\|_s < \|f - \mu_s(f|C_n)\|_s. \quad (4)$$

Applying once more the lemma of Fatou ($n \to \infty$), (4) and ($\ast$) imply

$$\left\| f - \lim_{n \in \mathbb{N}} \mu_s(f|C_n) \right\|_s < \lim_{n \in \mathbb{N}} \|f - \mu_s(f|C_n)\|_s = \|f - \mu_s(f|C_\infty)\|_s. \quad (5)$$

Since $\mu_s(f|C_n) \wedge \ldots \wedge \mu_s(f|C_m) \in C_m \subset C_\infty$ and $C_\infty$ is $\|\cdot\|_s$-closed, (4) and (5) imply that

$$\lim_{n \in \mathbb{N}} \mu_s(f|C_n) \in C_\infty.$$
Hence

\[ \lim_{n \to \infty} \mu_n(f|C_n) = \mu(f|C_\infty). \]

Using

\[
\|f - \mu_n(f|C_n) \lor \cdots \lor \mu_n(f|C_m)\|_s < \|f - \mu_n(f|C_n) \lor \cdots \lor \mu_n(f|C_{m-1})\|_s,
\]

which follows again from (1), one obtains completely analogously \( \lim_{n \to \infty} \mu_n(f|C_n) = \mu(f|C_\infty) \), which yields the assertion for the increasing case. If \( C_n \) is decreasing one obtains from (1) [instead of (2) and (2)*] for \( n < m \):

\[
\|f - \mu_n(f|C_n) \land \cdots \land \mu_n(f|C_m)\|_s < \|f - \mu_n(f|C_{n+1}) \land \cdots \land \mu_n(f|C_m)\|_s,
\]

\[
\|f - \mu_n(f|C_n) \lor \cdots \lor \mu_n(f|C_m)\|_s < \|f - \mu_n(f|C_{n+1}) \lor \cdots \lor \mu_n(f|C_m)\|_s.
\]

From (6) we obtain the assertion completely analogously as in the increasing case.

REFERENCES

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