ORTHOCOMPACTNESS AND PERFECT MAPPINGS

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Abstract. An example is given which shows that orthocompactness is not preserved by perfect maps. Subparacompact pointwise star-orthocompact spaces are orthocompact; this shows that orthocompactness is preserved by closed maps in the presence of subparacompactness.

A space $X$ is said to be orthocompact if every open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ such that if $\mathcal{V}' \subset \mathcal{V}$, then $\bigcap \mathcal{V}'$ is open in $X$. Such a refinement $\mathcal{V}$ of $\mathcal{U}$ is called a $Q$-refinement, and any open collection $\mathcal{W}$ such that $\bigcap \mathcal{W}'$ is open whenever $\mathcal{W}' \subset \mathcal{W}$ is called a $Q$-collection. The main purpose of this note is to provide an example showing the nonpreservation of orthocompactness under a perfect mapping, thus answering a question asked by B. Scott in [S4] and [S2]. The reader is referred to these papers for an in-depth discussion of orthocompactness, especially the product theory.

The description of the example follows below. We use the convention that an ordinal number is the set of smaller ordinals, and $I$ denotes the "closed unit interval" from $R$. A mapping is a continuous onto function.

Example 1. There exists an orthocompact space $X$ and a perfect mapping $f: X \rightarrow Y$ onto a nonorthocompact space $Y$.

Proof. Let $X_0 = \omega_1 \times I \times \{0\}$, $X_1 = \omega_1 \times I \times \{1\}$, and $X = X_0 \cup X_1$. For $\alpha, \beta \in \omega_1$, a nonlimit ordinal with $\alpha < \beta$, $x \in I$, and $\varepsilon > 0$ define

$$B(\alpha, \beta, x, \varepsilon) = \{(\gamma, z, 0) \in X_0: \alpha < \gamma < \beta, 0 < |x - z| < \varepsilon\}$$

$$\cup \{(\gamma, z, 1) \in X_1: \alpha < \gamma < \beta, |x - z| < \varepsilon\}.$$  

Topologize $X$ by describing local bases as follows: Points $(\beta, x, 0) \in X_0$ are isolated in $X$. Points $(\beta, x, 1) \in X_1$ have the set of all $B(\alpha, \beta, x, \varepsilon)$, for nonlimit $\alpha < \beta$ and $\varepsilon > 0$, for a local base. It may be revealing to the reader to provide a simple sketch here, and realize that $X$ is similar to, but not quite the same as, the "Alexandroff double" of $\omega_1 \times I$.

To show $X$ is orthocompact, let $\mathcal{U}$ be an open cover of $X$. For each $x \in I$ consider $\mathcal{U}$ as an open cover of $H_x = \omega_1 \times \{x\} \times \{1\}$. There exists a nonlimit ordinal $\alpha_x < \omega_1$, an uncountable subset $A_x \subset [\alpha_x, \omega_1)$, and $\varepsilon_x > 0$ (use $\varepsilon_x = 1/n$ for some appropriate positive integer $n$) such that for each $\beta \in A_x$ we have $B(\alpha_x, \beta, x, \varepsilon) \subset U$ for some $U \in \mathcal{U}$. Note that the collection $\mathcal{W}_x = \{B(\alpha_x, \beta, x, \varepsilon_x): \beta \in A_x\}$ is a $Q$-collection. For $x \in I$, let $J_x = \{z \in I: |x - z| < \varepsilon_x\}$; then $J_x \times \{x\}$ is an open cover of $I$ so there is a finite set $F \subset I$ such

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that \( I = \bigcup \{ J_x : x \in F \} \). If \( \beta_0 = \max \{ \alpha_x : x \in F \} \), the subspace
\[
Z = \{ (\alpha, x, i) : 0 < \alpha < \beta_0, x \in I, i \in \{0, 1\} \}
\]
is an open Lindelöf subspace of \( X \), so there is an open cover \( \mathcal{V} \) of \( Z \) such that \( \mathcal{V} \) is a \( Q \)-collection and each \( V \in \mathcal{V} \) is contained in some \( U \in \mathcal{U} \). It follows that
\[
\mathcal{W} = \mathcal{V} \cup \left( \bigcup \{ \{ x \} : x \in F \} \right) \cup \{ \{ p \} : p \in X_0 \}
\]
is a \( Q \)-cover of \( X \) that refines \( \mathcal{U} \).

Now let \( Y = X_0 \cup \omega_1 \) and define a map \( f : X \to Y : f(p) = p \) for \( p \in X_0 \) and \( f(\alpha, x, 1) = \alpha \) for \( (\alpha, x, 1) \in X_1 \). Let \( Y \) have the quotient topology induced by \( f \). Clearly \( f^{-1}(y) \) is compact in \( X \) for each \( y \in Y \) so to show \( f \) is perfect it suffices to show \( f \) is a closed mapping. Let \( E \subset X \) be closed; to show \( f(E) \) is closed in \( Y \) we need only show that for any \( \beta \in \omega_1 - f(E) \) there is an open neighborhood \( V \) of \( \beta \) in \( Y \) such that \( V \cap f(E) = \emptyset \). Now \( f^{-1}(\beta) \cap E = (\{ \beta \} \times I \times \{1\}) \cap E = \emptyset \) so for each \( x \in I \) there is \( \delta_x > 0 \) and nonlimit \( \alpha_x < \beta \) such that \( \{ \beta \} \times \{ x \} \times \{0\} \subset U \cap f(E) = \emptyset \).

Using the compactness of \( I \), we see that there is some \( r_0 > 0 \), nonlimit \( \gamma_0 < \beta \), and a finite set \( F \subset I \) such that
\[
\left( \bigcup \{ B(\gamma_0, \beta, x, r_0) : x \in F \} \right) \cap E = \emptyset
\]
and \( \{ \beta \} \times I \times \{1\} \cup \{ B(\gamma_0, \beta, x, r_0) : x \in F \} \) is saturated with respect to \( f \), we have \( V = f(\bigcup \{ B(\gamma_0, \beta, x, r_0) : x \in F \} ) \) as the desired neighborhood of \( \beta \) in \( Y \) where \( V \cap f(E) = \emptyset \).

To see that \( Y \) is not orthocompact, we note that if \( \beta \in \omega_1 \) and \( U \subset Y \) open, with \( \beta \in U \), there must be some nonlimit \( \alpha < \beta \) such that \( [\alpha, \beta] \subset U \) and \( [\alpha, \beta] \times \{ x \} \times \{0\} \subset U \) for all but finitely many \( x \in I \). Let \( B = \{ z_\alpha : \alpha < \omega_1 \} \) be a subset of \( I \), indexed by \( \omega_1 \), where \( z_\alpha \neq z_\beta \) if \( \alpha \neq \beta \). For each \( \beta < \omega_1 \) let \( G_\beta = f(B(0, \beta, z_\beta, 1)) \) and \( \mathcal{S} = \{ G_\beta : \beta < \omega_1 \} \); then \( \mathcal{S} \) is an open cover of \( Y \) and if \( \mathcal{K} \) is any open refinement of \( \mathcal{S} \) there is some \( \gamma \in \omega_1 \) such that \( [\gamma, \omega_1) \subset \text{St}(\gamma, \mathcal{K}) \). This can happen only if there is an uncountable set \( A \subset [\gamma, \omega_1) \) where for each \( \beta \in A \) there is \( H_\beta \in \mathcal{K} \) such that \( \gamma \in H_\beta \subset G_\beta \). It follows that
\[
\left( \bigcap_{\beta \in A} H_\beta \right) \cap \{ \omega_1 \times \{ z_\alpha \} \times \{0\} \} = \emptyset
\]
for every \( \alpha \in A \), hence \( \gamma \notin \text{int}(\bigcap_{\beta \in A} H_\beta) \) and \( \mathcal{K} \) cannot be a \( Q \)-refinement of \( \mathcal{S} \). That concludes the verification of the stated properties of Example 1.

The proof, in the above example, that \( Y \) is not orthocompact, was given for completeness. Other authors have considered similar examples and results which essentially show the nonorthocompactness of \( Y \). G. Gruenhage [G] gave an example of a nonorthocompact space which is the closed image of an orthocompact space. Gruenhage's range space is homeomorphic to a closed subspace of \( Y \) (and, under CH, is homeomorphic to \( Y \)) and certainly Gruenhage's result implies the nonorthocompactness of \( Y \). The essential reason for the nonorthocompactness of \( Y \) can also be culled from results in [S1] or [S2], which show that \( \omega_1 \times (\omega_1 + 1) \) is not orthocompact. For other related results on the construction of nonorthocompact spaces the reader is referred to [HL].
The existence of Example 1 increases the importance of several generalizations of orthocompactness, considered by other authors, that are preserved under closed or perfect mappings. Weakly orthocompact spaces [S1] are preserved under perfect maps, discretely orthocompact spaces [J] are preserved under closed mappings, and pointwise star-orthocompact spaces [G] are preserved under closed mappings. These concepts are useful in helping to preserve orthocompactness, under closed maps, when in the presence of other covering properties. Junnila [J] has shown that a \( \theta \)-refinable space \( X \) is orthocompact if it is discretely orthocompact (see [J] for definition) and as a corollary he obtains:

**THEOREM 2 [J].** If \( f: X \to Y \) is a closed continuous onto map, and \( X \) is a \( \theta \)-refinable orthocompact space, then so is \( Y \).

A somewhat weaker result can be obtained via the pointwise star-orthocompactness defined by Gruenhage [G]. A space \( X \) is pointwise star-orthocompact if for any open cover \( \mathcal{U} \) of \( X \) there is a \( Q \)-collection \( \{ V_x : x \in X \} \) such that \( x \in V_x \subseteq St(x, \mathcal{U}) \) for each \( x \in X \). Gruenhage shows that any pointwise star-orthocompact developable space is orthocompact; a modification of Gruenhage's proof actually yields the following stronger result.

**THEOREM 3.** If \( X \) is a subparacompact pointwise star-orthocompact space then \( X \) is orthocompact.

**Proof.** If \( \mathcal{U} \) is an open cover of the subparacompact space \( X \) there is a sequence \( \{ \mathcal{G}_n \} \) of open covers of \( X \) such that if \( x \in X \) there is some \( n \in N \) (depending on \( x \)) such that \( St(x, \mathcal{G}_n) \subseteq U \) for some \( U \in \mathcal{U} \) (see [B]). Apply pointwise star-orthocompactness to each \( \mathcal{G}_n \) and it follows that \( \mathcal{U} \) has an open refinement which is the union of a countable number of \( Q \)-collections. Since a subparacompact space is countably metacompact, we see that \( X \) is orthocompact [S1].

Since subparacompactness is preserved under closed maps [B], we have the weaker version of Theorem 2, using "subparacompact" in place of "\( \theta \)-refinable".

**REFERENCES**


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