ON THE HEIGHT OF THE FIRST STIEFEL-WHITNEY CLASS

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Abstract. Let $G_k(\mathbb{R}^{n+k})$ denote the Grassmann manifold of $k$-planes in real $(n + k)$-space and $w_1 \in H^1(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)$ the first Stiefel-Whitney class of the universal bundle. We determine, for many $(k, n)$, the exact height of $w_1$ in the cohomology ring. We also indicate the combinatorial significance of the complex analogue of these computations.

Let $G_k(\mathbb{R}^{n+k})$ denote the Grassmann manifold of $k$-planes in real $(n + k)$-space. We have previously determined estimates on the height of the first Stiefel-Whitney class $w_1$ of the universal $k$-plane bundle over $G_k(\mathbb{R}^{n+k})$ [2]. Indeed, we showed that if $1 < k < n$ and $2^r < n + k < 2^{r+1}$, then the height of $w_1$ is $2^r - \alpha$, where $\alpha$ is either 1 or 2 (unless $k = 2^r = n$, in which case we can only assert $1 < \alpha < 2^r$). As the codimension $n$ increases, it is clear that the height of $w_1$ is nondecreasing; hence it suffices to find where in the intervals of $n$: $[2^r + 1 - k, 2^{r+1} - k]$ the value of $\alpha$ jumps down. For $k = 2$, it was shown that $\alpha$ is identically 2 [2]. In this note, we settle the problem for $k = 3, 4, 5$ and for $k > 6$ (except for finitely many $n$).

We use the Schubert calculus description of the mod 2 cohomology of $G_k(\mathbb{R}^{n+k})$ [1]. Additively, a $\mathbb{Z}_2$-basis is provided by the Schubert symbols $(a_1, \ldots, a_k)$ where $0 < a_1 < \ldots < a_k < n$. Indeed, the tangential and normal Stiefel-Whitney classes of the universal bundle are themselves represented by Schubert symbols; namely

$$w_i = (0, \ldots, 0, 1, \ldots, 1), \quad w_j = (0, \ldots, 0, j),$$

where $1 < i < k$, $1 < j < n$. One often refers to the $w_j$ as special Schubert symbols. They are special, from one point of view, since there is a combinatorial formula describing their multiplication by an arbitrary Schubert symbol; namely

$$\tilde{w}_j(a_1, \ldots, a_k) = \sum (b_1, \ldots, b_k),$$

where the sum ranges over all $k$-tuples $(b_1, \ldots, b_k)$ such that $a_i < b_i < a_{i+1}$, $1 < i < k$ (where $a_{k+1} = n$) and $b_1 + \cdots + b_k = j + a_1 + \cdots + a_k$. This is called the Pieri intersection formula. We now begin with

**Lemma 1.** In $H^*(G_3(\mathbb{R}^{n+3}); \mathbb{Z}_2)$,

$$w_i^j = \sum_{p, q} \left[ \binom{p + q + 1}{q + 1} \binom{j + 1}{q + 1} \right] (p, q, j - p - q).$$

**Proof.** Use the Pieri formula and induction; see [3].
Lemma 2.

\[
\binom{2^r + j}{k} \equiv 0 \pmod{2} \quad \text{if } j + 1 < k < 2^r - 1.
\]

Proof. The result is immediate for \( j = 0 \); we then invoke induction and

\[
\binom{2^r + j}{k} = \binom{2^r + j - 1}{k} + \binom{2^r + j - 1}{k-1} \equiv 0 \pmod{2}
\]

for \( k > j + 1 \). This completes the proof.

It is now easy to show

Proposition 3. In \( H^*(G_3(\mathbb{R}^{2^r+2}); \mathbb{Z}_2) \),

\[
w_{i}^{2^{r+1}-1} = \sum_{i=1}^{2^{r}-1} (i, 2^r - i, 2^r - 1) \neq 0.
\]

Proof. By Lemma 1, this is equivalent to

\[
\binom{p + q - 1}{q} \equiv 0 \pmod{2}
\]

unless \( p + q = 2^r \) (for which clearly the binomial coefficient is odd). By the Schubert condition, \( q < 2^{r+1} - 1 - p - q \), i.e. \( q < 2^r - (p + 1/2) \). Also \( p + q > 2^r + 1 \), or \( q > 2^r - p + 1 \); so we can write \( q = 2^r - (p - 1) + j \), \( 0 < j < (p - 1/2) \). Reindexing by \( j \), we see

\[
\binom{p + q - 1}{q} = \binom{2^r + j}{2^r - (p - 1) + j}.
\]

By Lemma 2, it suffices to check that \( j + 1 < 2^r - (p - 1) + j < 2^r - 1 \). The first inequality becomes \( p < 2^r \) which is immediate from the Schubert condition. The second inequality asserts \( j + 2 < p \). So when \( j \) is maximized this becomes \( p > 3 \). This is also immediate, since if \( p < 3 \), it is easy to check that \( p + q = 2^r \). This completes the proof.

In [2, Lemma 4.6] we showed \( w_i^{2^{r+1}-1} = 0 \) in \( H^*(G_3(\mathbb{R}^{2^r+1}); \mathbb{Z}_2) \), so that for \( k = 3, \alpha = 2 \) at the first place in the interval and by Proposition 3, \( \alpha = 1 \) thereafter. To settle the case \( k = 4 \) we have

Proposition 4. For \( r > 3 \), in \( H^*(G_4(\mathbb{R}^{2^r+1}); \mathbb{Z}_2) \),

\[
w_{i}^{2^{r+1}-1} \neq 0.
\]

Proof. By Propositions 3.1 and 3.2 of [2], \( w_i^{2^r} = (1, 1, 1, 2^r - 3) \). So it suffices to compute the height of \( w_1 \) inside of \( H^*(G_3(\mathbb{R}^{2^r-1}); \mathbb{Z}_2) \). Now we have the inclusion

\[
i: G_3(\mathbb{R}^{2^r-1+2}) \subset G_3(\mathbb{R}^{2^r-1}).
\]

In the former space, by Proposition 3, \( w_1^{2^r-1} \neq 0 \); hence also in the latter space. It is easy to see there is an injection

\[
e: H^*(G_3(\mathbb{R}^{2^r-1}); \mathbb{Z}_2) \to H^*(G_4(\mathbb{R}^{2^r+1}); \mathbb{Z}_2)
\]
defined on the Schubert basis by

\[ e(a_1, a_2, a_3) = (a_1 + 1, a_2 + 1, a_3 + 1, 2^s - 3). \]

Graphically, we embed the 3-cycle in the upper left-hand corner

where the heights of the column indicate the entries of the Schubert cycle. The Pieri formula then implies

\[ w_1^{2s+2} = e(w_1^s) \]

where the locations of the \( w_1 \)'s are determined by context. Since \( e \) is injective and \( w_1^{2s+1} \neq 0 \), we conclude

\[ w_1^{2^s+1-1} = w_1^{2s+2} = e(w_1^{2s-1}) \neq 0. \]

Proposition 4 immediately yields that \( \alpha \) is identically 1 for the case \( k = 4 \).

**Proposition 5.** For \( s > 3 \), in \( H^*(G_4(R^{2^s+1}); Z_2) \),

\[ w_1^{2^s+1} \neq 0. \]

**Proof.** The argument is entirely analogous to that of Proposition 4, except we use the inclusion

\[ G_4(R^{2^s+1+1}) \subset G_4(R^{2^s+1}). \]

Hence, we must assume \( s > 4 \). For the case \( s = 3 \), we observe \( G_4(R^5) \cong G_4(R^5) \), so the result again follows from Proposition 4.

Finally, we come to

**Proposition 6.** For \( s > k - 2 \), in \( H^*(G_k(R^{2^s+1}); Z_2) \),

\[ w_1^{2^s+1} \neq 0. \]

**Proof.** Use the inclusion \( G_{k-1}(R^{2^{s-1}+1}) \subset G_{k-1}(R^{2^{s-1}}) \) and induction.

We summarize our results in the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( n )</th>
<th>( 2^s + 1 - k )</th>
<th>( 2^s + 2 - k )</th>
<th>\cdots</th>
<th>( 2^{s+1} - 1 - k )</th>
<th>( 2^{s+1} - k )</th>
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<td>( k = 2 )</td>
<td>( 2 )</td>
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<tr>
<td>( k = 3 )</td>
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<td>( k = 4 )</td>
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<td>( k = 5 )</td>
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<td>( k &gt; 6 )</td>
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\((s > k - 2)\)

Values of \( \alpha(k, n) \), where \( k < n \), i.e. \( k < 2^s - 1 \); height of \( w_1 = 2^{s+1} - \alpha(k, n) \) in the mod 2 cohomology of \( G_k(R^{s+k}) \).
Remark. One can ask similar questions for the first Chern class $c_1$ in $H^2(G_k(\mathbb{C}^{n+k}); \mathbb{Z})$. Of course, the integral version of the Pieri formula immediately implies that $c_1^{nk} \neq 0$. (One could also argue that $c_1$ is a Kähler 2-form, so raising it to the complex dimension of the manifold is a volume element.) Now the $2nk$-cohomology group of $\mathbb{C}^{n+k}$ is generated by the fundamental class $(n, n, \ldots, n)$; so $c_1^{nk} = N(n, \ldots, n)$ for some integer $N$. It is an immediate consequence of the “hook formula” that

$$N = \frac{(nk)!}{1!2! \cdots k!(k + 1)! \cdots n!(n + 1)! \cdots (n + k - 1)!}.$$ 

The number on the right can also be interpreted as the degree of a certain irreducible representation of the symmetric group on $nk$ letters. It might be interesting to try a similar computation for $G/P$, where $G$ is a complex Lie group and $P$ is a maximal parabolic.

Finally, note that if $k = 2$, $N$ is the $n$th Catalan number $(n + 1)^{-1} \binom{2n}{n}$.

References


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