THE ZYGMUND CONDITION FOR BLOCH FUNCTIONS
IN THE BALL IN $\mathbb{C}^n$

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Abstract. In this paper we prove the equivalence of the Bloch condition for a holomorphic function $f$ on the ball $B_n$ with the Zygmund second difference condition for a suitable primitive $F$ of $f$.

Introduction. If $B_1$ is the open unit disc in the complex plane and $f$ is holomorphic on $B_1$, we say that $f$ is a Bloch function if there exists a positive number $M$ such that

$$|f'(z)|(1 - |z|^2) < M$$

for all $z \in B_1$. When equipped with an appropriate norm the linear space of Bloch functions becomes a nonseparable Banach space. There are many equivalent conditions that a function can satisfy to be a Bloch function (see Pommerenke [2] or Cima [1]). In a recent thesis, Richard Timoney [4] has done an exhaustive study of properties of Bloch functions on domains in $\mathbb{C}^n$. In particular his work includes the theory of Bloch functions on the ball

$$B_n = \{ z \in \mathbb{C}^n : \|z\| = \sqrt{|z|^2} < 1 \}.$$ 

He has shown that all the known characterizations, save two, that are equivalent for the case of $n = 1$ are valid for $n > 1$. One of these two characterizations is the second difference condition of Zygmund [5]. We will establish the equivalence of this condition for the $B_n$ case.

1. Preliminaries. Assume $f$ is a holomorphic function of $B_n \to \mathbb{C}$. For $u$ and $v$ vectors in $\mathbb{C}^n$, $z \in B_n$ and $\langle u, v \rangle = \sum_{j=1}^n u_j \bar{v}_j$ the Bergman metric is given by

$$H_z(u, \bar{v}) = \left( \frac{n+1}{2} \right) \frac{(1 - \|z\|^2)\langle u, \bar{v} \rangle + \langle u, \bar{x} \rangle \langle z, \bar{v} \rangle}{(1 - \|z\|^2)^2}.$$ 

For each $z \in B_n$ define

$$Q_f(z) = \sup \{ |(\nabla_x f)(x)| / H_x(x, \bar{x})^{1/2} ; x \in C^n, x \neq 0 \}$$

where $\langle \nabla_x f \rangle(x) = \sum_{j=1}^n (\partial f / \partial z_j)(x) x_j$. 

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Definition 1. A holomorphic function \( f: B_1 \rightarrow \mathbb{C} \) is called a Bloch function if
\[
\sup \{ \frac{1}{|z|} : z \in B_1 \} < \infty.
\]

In considering this definition, a certain amount of pathology immediately enters. In fact one observes by studying the metric that if \( f \) is a Bloch function, then the growth of its directional derivative in the radial direction is \( O((1 - \|z\|^2)^{-1}) \), whereas the growth in directions orthogonal to the radial direction is only \( O((1 - \|z\|^2)^{-1/2}) \). Timoney \([4]\) shows that the Bloch condition is equivalent to the following condition:
\[
\sup \{ \| \nabla f \| (1 - \|z\|^2) : z \in B_1 \} < \infty. \tag{1.1}
\]

Assume \( F \) is in the ball algebra of \( B_1 \), i.e., \( F \) is continuous on \( \overline{B}_1 \) and holomorphic in \( B_1 \). Further assume that the boundary values of \( F \) satisfy
\[
|F(e^{i(\theta + h)}) + F(e^{i(\theta - h)}) - 2F(e^{i\theta})| < A|h| \tag{1.2}
\]
for all real numbers \( h \) and some positive number \( A \), independent of \( \theta \). In \([5]\) it was shown that a function \( f \), holomorphic on \( B_1 \), satisfies the Bloch condition if and only if its primitive \( F(z) = \int_0^z f(\zeta) \, d\zeta \) is in the ball algebra of \( B_1 \) and satisfies condition (1.2).

We consider in this note \( C^1 \) curves \( \gamma \) mapping \( \mathbb{R} \rightarrow \partial B_1 \) such that \( |\gamma'(t)| = 1 \) for all \( t \). We refer to these as normalized \( C^1 \) curves. With this class of curves in mind we make the following definition.

Definition 1.2. Let \( F \) be in the ball algebra of \( B_1 \). We say that \( F \) satisfies condition \( \Lambda_A(\partial B_n) \) if there exists a positive number \( A \) such that for all normalized \( C^1 \) curves \( \gamma \) in \( B_1 \),
\[
|F(\gamma(t + h)) + F(\gamma(t - h)) - 2F(\gamma(t))| < A|h| (2.1)
\]
for all \( t \) and \( h \) in \( \mathbb{R} \).

Finally, if \( f \) is holomorphic on \( B_n \) with expansion in terms of homogeneous polynomials given by \( f(z) = \sum_{k=0}^{\infty} F_k(z) \) define the radial derivative \( \Re f \) of \( f \) by the formula
\[
\Re f(z) = \sum_{k=1}^{\infty} kF_k(z).
\]

2. The principal result. If we are given a function \( f \) holomorphic on \( B_n \), define a function
\[
F(z) = (\Re f)(z) \equiv \int_0^1 f(tz) \, dt.
\]
One checks that \( \Re (\Re f)(z) = \Re F(z) = f(z) - F(z) \).

Theorem 1. A function \( f \) holomorphic on \( B_n \) is in the Bloch space of \( B_n \) if and only if \( \Re f \) satisfies the \( \Lambda_A(\partial B_n) \) condition.

Proof. Assume first that \( f \) is a Bloch function. For \( a \in \partial B_n \) the slice functions are defined by \( f_\alpha(\lambda) = f(\lambda a) \), \( \lambda \in B_1 \). Since
\[
(\Re f)_\alpha(\lambda) = F_\alpha(\lambda) = \frac{1}{\lambda} \int_0^\lambda f_\alpha(\zeta) \, d\zeta \tag{2.1}
\]
we see that each $F_a$ is in $\Lambda_\alpha(\partial B_1)$. Also each member of the family $\{F_a; a \in \partial B_n\}$ has its oscillation $\omega(F_a, \delta) = O(\delta \log \delta)$, uniformly in $a$. $F$ can be extended to a function on $\overline{B}_n$ by using the values on the slices. Now with $b \in \partial B_n$, $0 < r < 1$,

$$|F(a) - F(b)| < M(1 - r) \log (1 - r) + |F_a(r) - F_b(r)|.$$  

This shows that $F$ is continuous at $a$ and hence is in the ball algebra.

Now fix a normalized $C^1$ curve $\gamma$ with range in $\partial B_n$ and let $h > 0$ be given. If $g(t)$ is any function defined on $\mathbb{R}$ set $Fg(t) = g(t + h) - g(t)$. With $r = 1 - h$ we write

$$F(\gamma(t)) = (F(\gamma(t)) - F(\gamma(t))) + (F(\gamma(t)))$$

and show that $A^2$ of each expression in parentheses is $O(h)$, independent of $t$. Since $|f(z)| = O(\log(1 - |z|))$ one easily verifies that

$$F(\gamma(t)) - F(\gamma(t)) = (1 - r)f(\gamma(t)) + \int_0^1 (1 - s) \nabla f(\gamma(s)) \cdot \gamma(s) \, ds.$$  

The integral in this equality is $O(h)$. Also

$$Fg(\gamma(t)) = \int_0^h \nabla f(\gamma(t)) \cdot \gamma'(t) = O(1)$$

where $\gamma(0) = \gamma(t + p)$. Hence

$$A^2[F(\gamma(t)) - F(\gamma(t))] = O(h)$$

uniformly in $t$. The expression $A^2F(\gamma(t))$ involves three terms:

$$r \int_0^h (\gamma'(t + p) - \gamma'(t)) \cdot \nabla F(\gamma(t + p)) \, \phi, \quad (2.2)$$

$$r \int_0^h \gamma'(t) \cdot (\nabla F(\gamma(t + p)) - \nabla F(\gamma(t - p))) \, \phi, \quad (2.3)$$

$$r \int_0^h (\gamma'(t) - \gamma'(t - p)) \cdot \nabla F(\gamma(t - p)) \, \phi. \quad (2.4)$$

By the definition

$$\left| \frac{\partial F}{\partial z_j}(\gamma(t)) \right| = r \int_0^1 \nabla f(\gamma(t)) \cdot \gamma'(t) \, du \leq M \cdot |\log h|.$$  

Thus, expressions (2.2) and (2.4) are $O(1)$ uniformly in $t$. Similarly

$$\|\nabla F(\gamma(t)) - \nabla F(\gamma(s))\| = O(\log(1 - r))$$

and hence (2.3) is $O(1)$.

For the converse we observe that for each $0 < \alpha < 1$, the space $\Lambda_\alpha(B_1)$ of functions in the ball algebra with boundary values in the Lip $\alpha$ space contains $\Lambda_* (B_1)$. Further, bounded subsets of $\Lambda_\alpha$ are bounded in $\Lambda_*$. Fix $z = re_1$ in $B_n$ with $\|e_1\| = 1$, $0 < r < 1$, and let $\{e_j\}_{j=1}^n$ be an orthonormal basis for $C^n$. Let $D_j$ be the derivative in the $e_j$ direction. We can apply a result of Rudin [3] to draw the following conclusions. Since $\{F_w\}$ is a norm-bounded subset of $\Lambda_{1/2}(B_1)$,

$$(\Re F)(z) = O((1 - \|z\|)^{-1/2})$$

and

$$D_j f(z) = O((1 - \|z\|)^{-1}), \quad 2 < j < n.$$
Since \( F_a (r) + (\mathcal{R} F)_a (r) = f_a (r) \) we apply the one variable result to conclude
\[
D_M = D_{f_a} (r) = O((1 - ||z||)^{-1}).
\]
The estimates are uniform.

The referee has pointed out that our proof yields the following equivalences.

**Proposition 1.** A homomorphic function \( f : B_n \to C \) is a Bloch function if and only if the slice functions \( F_a = (\mathcal{F} f)_a, a \in \partial B_n, \) are uniformly bounded in \( \Lambda (\partial B_1) \).

**Proof.** From [4] a function \( f : B_n \to C \) is a Bloch function if and only if
\[
\sup_{z \in B_n} |(\mathcal{F} f)(z)|(1 - ||z||^2) < \infty. \tag{2.5}
\]
The functions \( (\mathcal{F} f)_a, a \in \partial B_n, \) are uniformly bounded in \( \Lambda (\partial B_1) \) if and only if
\[
\sup_{|z| < 1, a \in \partial B_n} |[(\mathcal{F} f)_a]'' (z)|(1 - |z|^2) < \infty. \tag{2.6}
\]
A computation with (2.1) shows that
\[
[(\mathcal{F} f)_a]'' (z) = \frac{1}{z^2} (\mathcal{F} f)_a (z) - \frac{2}{z} [f_a (z) - (\mathcal{F} f)_a (z)]
\]
\[
= \frac{1}{z^2} (\mathcal{F} f)_a (z) - \frac{2}{z} [(\mathcal{F} f)_a]'' (z).
\]
It is clear that (2.5) and (2.6) are equivalent.

**Proposition 2.** A holomorphic function \( F : B_n \to C \) is in \( \Lambda (\partial B_n) \) if and only if the slice functions \( F_a, a \in \partial B_n, \) are uniformly bounded in \( \Lambda (\partial B_1) \).

**Proof.** This follows from Rudin’s result [3] and the proof of Theorem 1.

A comment is in order. If one considers the latter half of the proof of Theorem 1, one sees that \( \mathcal{F} f = F \) is much more smooth on curves \( \gamma (t) \) whose tangents lie in the “complex tangential direction.” However, the function \( f(z) = \log(1 - z_1^2 - z_2^2) \) achieves the proper growth estimate on curves \( \gamma (t) = e^{itw} \) \((||w|| = 1)\) whose tangents lie in the real direction.

References


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