A NORMAL FORM FOR A SPECIAL CLASS OF CURVATURE OPERATORS

STANLEY M. ZOLTEK

Abstract. In the case of a 4-dimensional oriented inner product space Singer and Thorpe found a canonical form for a curvature operator which commutes with a generator of $\Lambda^4$, and used it to prove that the curvature function is completely determined by its critical point behavior. In dimension 5 we extend these results to curvature operators which commute with an element of $\Lambda^4$.

1. Introduction. The Riemannian sectional curvature of a Riemannian manifold $M$ is a real valued function $\sigma$ on the Grassmann bundle of tangent 2-planes of $M$. Although there exist many theorems relating the curvature of $M$ to various topological and geometric properties of $M$, there is little known of a general nature about the behavior of $\sigma$ itself.

In [3], Thorpe gave a simple characterization of the minimal and maximal sets of $\sigma$, when the dimension $M < 4$. However his characterization does not hold in higher dimensions, see [4].

Let $V$ be a real $n$-dimensional vector space with inner product, $\langle \cdot, \cdot \rangle$. It is well known that if $T$ is a symmetric linear transformation on $V$, then the diagonalization or normal form of $T$ is equivalent to the analysis of the critical point behavior of the function $v \mapsto \langle Tv, v \rangle$ on the projective space of $V$. It is also known, [1], that the curvature tensor at a point of a Riemannian manifold is completely determined by the sectional curvature function $\sigma$ on the Grassmann manifold of 2-planes at the point. It then seems natural to look for a "normal form" for the curvature tensor by analyzing the critical point behavior of the sectional curvature function $\sigma$.

In [2] Singer and Thorpe prove that the curvature tensor of a 4-dimensional oriented Einstein manifold is completely determined by the critical point behavior of $\sigma$. Working at a point $p$, of a 4-dimensional oriented Riemannian manifold $M$, they view the curvature as a symmetric linear operator on $\Lambda^2$ of the tangent space $TM_p$. When the operator commutes with a generator of $\Lambda^4(TM_p)$, they show it has a simple canonical form and use it to prove that the critical point behavior of $\sigma$ determines the operator. In dimension 5 we extend these results to curvature operators which commute with an element of $\Lambda^4$; the curvature tensor of the 5 sphere or of the product of an Einstein space and a line are examples of such operators.

Received by the editors April 15, 1979 and, in revised form, September 14, 1979; presented to the Society, January 25, 1979.


Key words and phrases. Curvature operator, normal form, critical point behavior, Einstein manifold, Bianchi identity, Grassmann quadratic 2-relations.

© 1980 American Mathematical Society 0002-9939/80/0000-0374/02.25

614

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
2. Preliminaries. Let $V$ be an $n$-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$ and for $v \in V$ set $|v| = \sqrt{\langle v, v \rangle}$. For $p$ an integer, $1 \leq p \leq n$, by $\Lambda^p(V)$ or $\Lambda^p$ we mean the space of $p$-vectors of $V$. If $\{e_1, \ldots, e_n\}$ is a basis for $V$, then $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leq i_1 < \cdots < i_p \}$ is a basis for $\Lambda^p$ and it follows that $\Lambda^p$ has dimension $\binom{n}{p}$. A $p$-vector $w$ is called decomposable if $w = v_1 \wedge \cdots \wedge v_p$ where $v_1, \ldots, v_p \in V$. Hence, $\Lambda^p$ has a basis of decomposable vectors. Thus, when defining an inner product on $\Lambda^p$ it suffices to specify its values on decomposable $p$-vectors. We set $<w, A \cdot \cdots \cdot A v\cdot > = \det \langle u, v \rangle$ where $u, v \in V$. For $e \in \Lambda^2$ we set $||e|| = \sqrt{\langle e, e \rangle}$. It follows that if $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $V$, then $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leq i_1 < \cdots < i_p \}$ is an orthonormal basis for $\Lambda^p$. Let $G$ denote the Grassmann manifold of oriented 2-dimensional subspaces of $V$; we identify $G$ with the submanifold of $\Lambda^2$ consisting of decomposable 2-vectors of length 1 by $P \to u \wedge v$ where $\{u, v\}$ is any oriented orthonormal basis for $P$.

Let $V$ be an $n$-dimensional real inner product space. A curvature operator $R$ is a symmetric linear transformation on $\Lambda^2(V)$. The space $\mathcal{S}$ of all curvature operators has dimension $\{(n^2 + n)/2 \}$ and inner product given by $\langle R, T \rangle = \text{trace } R \circ T$ where $R, T \in \mathcal{S}$. Given $R \in \mathcal{S}$, its sectional curvature is the function $\sigma_R: G \to \mathbb{R}$ defined by $\sigma_R(P) = \langle RP, P \rangle$, $P \in G$. A 2-plane (2-vector) $P$ is a critical plane of $R$ if $P$ is a critical point of $\sigma_R$.

3. The Bianchi identity and the Grassmann quadratic 2-relations. In this section we examine the space $\mathcal{S}$ which is complementary in $\mathcal{S}$ to the subspace $\mathcal{S}_\mathcal{B} = \{R \in \mathcal{S} : R \text{ satisfies the Bianchi identity} \}$. We recall that $\mathcal{S}$ is naturally isomorphic to $\Lambda^4$ and we exhibit the relationship between $\mathcal{S}$ and the Grassmann quadratic 2-relations which are necessary and sufficient conditions for decomposability of elements in $\Lambda^2$. These results are well known and detailed proofs can be found in [3].

Given $R \in \mathcal{S}$ we associate a 2-form on $V$ with values in the vector space of skew symmetric endomorphisms of $V$ by $\langle R(u, v)(w), x \rangle = \langle Ru \wedge v, w \wedge x \rangle$, $u, v, w, x \in V$. It is easily checked that this "association" is a vector space isomorphism.

Using this identification we define the Bianchi map $b: \mathcal{S} \to \mathcal{S}$; given $R \in \mathcal{S}$ we set $[b(R)](u, v)(w) = R(u, v)(w) + R(v, w)(u) + R(w, u)(v)$. It is easily checked that $b$ is a linear map and so its kernel is a linear subspace of $\mathcal{S}$ which we denote by $\mathcal{S}_\mathcal{B}$. Let $\mathcal{S} = \mathcal{B}^\perp$, the orthogonal complement of $\mathcal{B}$ in $\mathcal{S}$. For each $e \in \Lambda^4$ we associate $S_e \in \mathcal{S}$ by $\langle S_e \alpha, \beta \rangle = \langle e, \alpha \wedge \beta \rangle$, where $\alpha, \beta \in \Lambda^2$.

**Proposition 3.1.** The map $e \to S_e$ is an isomorphism of $\Lambda^4$ onto $\mathcal{S}$. In fact $e \to S_e/\sqrt{6}$ is an isometry.

**Proposition 3.2.** Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $V$. For $1 \leq i < j < k < l < n$ set $S_{ijkl} = S_{e_i \wedge e_j \wedge e_k \wedge e_l}$. If $\alpha \in \Lambda^2$, then $\alpha$ is decomposable if and only if $\langle S_{ijkl} \alpha, \alpha \rangle = 0$, $1 \leq i < j < k < l < n$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Remark 1. The conditions $\langle S_{ijk}\alpha, \alpha \rangle = 0$, $1 < i < j < k < l < n$, are known as the Grassmann quadratic 2-relations.

Remark 2. In view of Proposition 3.2 it is clear that each curvature operator $S \in \mathfrak{S}$ has sectional curvature $\sigma_S$ identically zero. Conversely, it is easily checked that this property characterizes $\mathfrak{S}$.

4. Critical planes and normal forms. In this section we restrict ourselves to the case where dimension $V = 5$. Letting $\{e_1, \ldots, e_5\}$ be an orthonormal basis for $V$ we use the operators $S_{ijk}$, $1 < i < j < k < l < 5$, defined in §3, to derive necessary and sufficient conditions for $P \in G$ to be a critical plane of a curvature operator $R$. Next we derive a normal form for a special class of curvature operators and use this normal form to prove that these operators are determined uniquely by their critical planes and values of sectional curvature on these critical planes.

Set $J = \{(i, j, k, l) : 1 < i < j < k < l < 5\}$.

Proposition 4.1. $P \in G$ is a critical plane of $R$ if and only if

$$RP = \lambda P + \sum J \mu_{ijkl} S_{ijkl} P$$

for some real numbers $\lambda$ and $\mu_{ijkl}$, $(i, j, k, l) \in J$. The number $\lambda$ is the (critical) value of $\sigma_R$ at $P$.

Proof. Consider the real valued functions defined on $\Lambda^2$ by $f(\xi) = \langle R \xi, \xi \rangle$, $g(\xi) = \langle \xi, \xi \rangle$, and $h_{ijkl}(\xi) = \langle S_{ijkl} \xi, \xi \rangle$, $1 < i < j < k < l < 5$. Then $\sigma$ is the restriction of $f$ to $G = g^{-1}(1) \cap h_{ijkl}^{-1}(0)$. $P \in G$ is a critical point of $\sigma_R$ if and only if at $P$ we have

$$\nabla f = \lambda \nabla g + \sum J \mu_{ijkl} S_{ijkl} P$$

where $\lambda$ and $\mu_{ijkl}$, $(i, j, k, l) \in J$, are Lagrange multipliers. Now for a quadratic form $\langle A \xi, \xi \rangle$, the gradient at $P$ is just $2AP$. Applying this to the quadratic forms $f$, $g$, and $h_{ijkl}$, $1 < i < j < k < l < 5$, completes the proof.

Theorem 4.2. Let $R \in \mathfrak{R}$ and suppose that for some $\eta \in \Lambda^4$, $\eta \neq 0$, $RS_\eta = S_\eta R$. Then there exists an orthonormal basis for $\Lambda^2$ consisting of critical planes of $R$ and relative to this basis the matrix for $R$ has the form

$$[R] = \begin{bmatrix} A & B & 0 \\ B & A & 0 \\ 0 & 0 & C \end{bmatrix}$$

where

$$A = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}, \quad B = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

and

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. We note that since dimension $V = 5$ each element of $\Lambda^4(V)$ is decomposable. Let $\{e_1, \ldots, e_5\}$ be an orthonormal basis for $V$ such that $\eta/\|\eta\| = e_1 \wedge e_2 \wedge e_3$, since $RS_\eta = S_\eta R$ the symmetric operators $R$ and $S_\eta$ can be simultaneously diagonalized. Hence we can let $\{\xi_1, \ldots, \xi_{10}\}$ be an orthonormal basis for $\Lambda^2$ such that $S_\eta \xi_i = \xi_i$ ($1 < i < 3$), $S_\eta \xi_i = -\xi_i$ ($4 < i < 6$), $S_\eta \xi_i = 0$ ($7 < i < 10$), and $R \xi_i = a_i \xi_i$ ($1 < i < 10$). It is easily checked that $\{e_1 \wedge e_5, e_2 \wedge e_3, e_4 \wedge e_5\}$ is a basis for the zero eigenspace of $S_\eta$ and then that each eigenvector with eigenvalue zero (in particular $\xi_7, \xi_8, \xi_9$, and $\xi_{10}$) is decomposable.

Let $W = \text{Span}(e_1, e_2, e_3, e_4)$ and consider

$$P_i = \begin{cases} (\xi_i + \xi_{i+3})/\sqrt{2}, & 1 < i < 3, \\ (\xi_{i-3} - \xi_i)/\sqrt{2}, & 4 < i < 6. \end{cases}$$

It is readily checked that $\langle S_\eta P_i, P_i \rangle = 0$, and since $P_i \in \Lambda^2(W)$, $1 < i < 6$, it follows by Proposition 3.2 that each $P_i$ is decomposable. Now $R \xi_i = a_i \xi_i$ ($7 < i < 10$) and a short computation shows that $RP_i = \lambda_i P_i + \mu_i S_\eta P_i$, $1 < i < 6$, where $\lambda_i = \lambda_{i+3} = (a_i + a_{i+3})/2$, $1 < i < 3$, and $\mu_i = \mu_{i+3} = (a_i - a_{i+3})/2$, $1 < i < 3$. Hence it follows by Proposition 4.1 that $\{P_1, P_2, P_3, P_4, P_5, P_6, \xi_7, \xi_8, \xi_9, \xi_{10}\}$ is an orthonormal basis for $\Lambda^2$ consisting of critical planes of $R$. The matrix $[R]$ with respect to this basis has the required form.

Remark 4.3. The above normal form is not unique. In fact an examination of the proof of Theorem 4.2 shows that (neglecting orientations) when the eigenvalues of $R$ are distinct there are nine distinct critical plane pairs $(P_{ij}, P_{kl})$ one corresponding to each pair $(\xi_i, \xi_{i+3})$ ($1 < i < 3$) of eigenvectors of $R$. Since $P_{ij}^\pm$ and $P_{kl}^\pm$ are orthogonal if and only if $i \neq k$ and $j \neq l$, it follows that there are six distinct normal forms. (We make no distinction between two matrices which differ by permutations of rows and columns.)

When the eigenvalues of $R$ are not distinct, then the critical values of $\sigma$ may not be distinct and we may get less than six distinct normal forms.

Theorem 4.4. Suppose $R \in B$ and that for some $\eta \in \Lambda^4, \eta \neq 0$, $RS_\eta = S_\eta R$. Then $R$ is completely determined by its critical planes and the values of the sectional curvature on these planes.

Proof. Let $P_{ij}^\pm$ ($i, j < 3$) and $\xi_i$ ($6 < i < 10$) be the critical planes in the proof of Theorem 4.2 and Remark 4.3. Let $\lambda_{i, j} = \sigma(P_{ij}^\pm)$. Then $RP_{ij}^\pm = \lambda_{ij}^\pm P_{ij}^\pm + \mu_{ij} S_\eta P_{ij}^\pm$ where $\lambda_{ij}^\pm = (|a_i + a_{i+3}|)/2$, $|a_i - a_{i+3}|/2$. Since $R \in B, 0 = \langle R, S_\eta \rangle = RS_\eta = a_1 + a_2 + a_3 - a_4 - a_5 - a_6$ and a short computation shows that $\mu_{ij} = (\lambda_{ij} + \lambda_{ij'} - \lambda_{ij''})/3$ where $(i, i', i'')$ and $(j, j', j'')$ are arbitrary permutations of $(1, 2, 3)$. Thus, the matrix for $R$ (with respect to the basis $\{P_{11}^+, P_{22}^+, P_{33}^+, P_{11}, P_{22}, P_{33}, \xi_7, \xi_8, \xi_9, \xi_{10}\}$) and so $R$ itself is completely determined by the $\lambda_{ij}$'s.
REFERENCES


Department of Mathematics, Wright State University, Dayton, Ohio 45435