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ULTRAPRODUCT INVARIANT LOGICS

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ABSTRACT. In this paper we give a construction of logics via the property of being preserved from the models to their ultraproduct. Specific examples are given which include some cardinality quantifiers.

We will assume that the reader is familiar with basic model theory including ultraproducts, Chang and Keisler [4], and abstract logics, confer Barwise [1].

DEFINITION. $(\mathfrak{A}, \vec{q})$ is called a *generalized model* if $\vec{q}(i) \subseteq \mathcal{P}(A^{m(i)})$ for each i (where $m(i) \in \omega$ and the function m is called the type of the model).

Take Mod to be a class of generalized models of the same type closed under restriction and expansion of languages.

Let $\mathrm{Seq}_\kappa(\mathrm{Mod})$ be the class of all sequences of length κ of elements of Mod, $\mathrm{Ult}(\kappa)$ be the set of ultrafilters on κ, and $\mathrm{Seq}(\mathrm{Mod}) = \bigcup_\kappa \mathrm{Seq}_\kappa(\mathrm{Mod})$ and $\mathrm{Ult} = \bigcup_\kappa \mathrm{Ult}(\kappa)$.

DEFINITION. An *ultraproduct on* Mod is a partial function ULT from $\bigcup_\kappa \mathrm{Seq}_\kappa(\mathrm{Mod}) \times \mathrm{Ult}(\kappa)$ into Mod.

We say that the pair $\langle \mathrm{Mod}, \mathrm{ULT}\rangle$ is closed under κ-ultraproducts if $\mathrm{Seq}_\kappa(\mathrm{Mod}) \times \mathrm{Ult}(\kappa) \subseteq \mathrm{Dom}\ \mathrm{ULT}$. If $\langle \mathrm{Mod}, \mathrm{ULT}\rangle$ is closed under κ-ultraproducts for all cardinals κ then it is said to be closed under ultraproducts.

Let $\langle \mathfrak{L}^2, \models_{\mathfrak{L}^2}\rangle$ be the second-order logic with additional quantification over the generalized domains. That is:

(a) the usual clauses for generalized first-order logics,

(b) if t is a term and $X_m^{\vec{q}(i)}$ is a generalized variable then $t \in X_m^{\vec{q}(i)}$ is a formula and $(\mathfrak{A}, \vec{q}) \models_{\mathfrak{L}^2} t \in X_m^{\vec{q}(i)}[s]$ if and only if $s(t) \in s(X_m^{\vec{q}(i)})$ where $s(X_m^{\vec{q}(i)}) \in \vec{q}(i)$,

(c) if $\varphi(R)$ (or $\varphi(f)$) are formulas where R (f) is a relation (function) variable then $\exists R\varphi(R)$ ($\exists f\varphi(f)$) is a formula. $\exists R\varphi(R)$ and $\exists f\varphi(f)$ have the usual second-order interpretation,

(d) if $\varphi(X^{\vec{q}(i)})$ is a formula then so is $\exists X_m^{\vec{q}(i)}\varphi(X_m^{\vec{q}(i)})$. $(\mathfrak{A}, \vec{q}) \models_{\mathfrak{L}^2} \exists X_m^{\vec{q}(i)}\varphi(X_m^{\vec{q}(i)})$ if and only if there is an assignment s such that $(\mathfrak{A}, \vec{q}) \models_{\mathfrak{L}^2} \varphi(X_m^{\vec{q}(i)})[s]$.

Let Mod and ULT be given.

DEFINITION. A sentence φ in the language L of $\mathfrak{L}^2$ is called *invariant* if and only if for each $\langle\{(\mathfrak{A}_\alpha, \vec{q}_\alpha)\}_{\alpha<\kappa}, D\rangle \in \mathrm{Dom}\ \mathrm{ULT}$ (each $(\mathfrak{A}_\alpha, q_\alpha)$ is an L-model) $\{\alpha | (\mathfrak{A}_\alpha, q_\alpha) \models_{\mathfrak{L}^2} \varphi\} \in D$ if and only if $\mathrm{ULT}(\{(\mathfrak{A}_\alpha, q_\alpha)\}_{\alpha\in\kappa}, D) \models_{\mathfrak{L}^2} \varphi$.

Let $I(\mathrm{Mod}, \mathrm{ULT})$ be the sublogic of $\mathfrak{L}^2$ consisting of the invariant sentences.

Let $\mathrm{Ult}^*(\kappa)$ be the class of regular ultrafilters over κ.

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THEOREM. *If* $\mathrm{Seq}_\kappa(\mathrm{Mod}) \times \mathrm{Ult}^*(\kappa) \subseteq \mathrm{Dom\ ULT}$ *then* $I(\mathrm{Mod}, \mathrm{ULT})$ *is* (ω, κ)-*compact. In fact it satisfies a* κ *version of the Łoś ultraproducts theorem.*

PROOF. Straightforward by the definition of $I(\mathrm{Mod}, \mathrm{ULT})$.

THEOREM. *Let* ULT *commute with restriction. Then* $I(\mathrm{Mod}, \mathrm{ULT})$ *has the Souslin-Kleene Property* (*i.e. complementary* $PC(L)$ *classes are* $EC(L)$ *classes*).

REMARK. For ULT to commute with restriction we mean that if $\mathrm{ULT}(\{(\mathfrak{A}_\alpha, q_\alpha)\}, D) = (\mathfrak{B}, r)$ then $(\mathfrak{B} \restriction L, r) = \mathrm{ULT}(\{(\mathfrak{A}_\alpha \restriction L, q_\alpha)\}, D)$.

PROOF. Let θ_0 be a sentence of L_0, θ_1 of L_1,

$$\Omega = \{(\mathfrak{A} \restriction L, \vec{q}) | (\mathfrak{A}, \vec{q}) \models \theta_0\},$$
$$\Omega^c = \{(\mathfrak{A} \restriction L, \vec{q}) | (\mathfrak{A}, q) \models \theta_1\}.$$

We need to show that Ω is $EC(L_0 \cap L_1)$.

Define $\vec{\exists}\theta_i$ to be the sentence obtained from θ_i by replacing all the relation and function symbols of $L_i - L_0 \cap L_1$ by variables and existentially quantifying over them.

We claim that $\vec{\exists}\theta_i$ is an $L_0 \cap L_1$-invariant sentence. Suppose $\{\alpha | (\mathfrak{A}_\alpha, q_\alpha) \models \vec{\exists}\theta_i\} \in D$. Then there are $\vec{R}_\alpha$, $\vec{f}_\alpha$ such that $\{\alpha | (\mathfrak{A}_\alpha, \vec{R}_\alpha, \vec{f}_\alpha, \vec{q}) \models \theta_i\} \in D$. Hence $\mathrm{ULT}(\{(\mathfrak{A}_\alpha, \vec{R}_\alpha, \vec{f}_\alpha, \vec{q}_\alpha)\}_{\alpha\in\kappa}, D) \models \theta_i$.

Using commutativity we get that $\mathrm{ULT}(\{(\mathfrak{A}_\alpha, q_\alpha)\}, D) \models \vec{\exists}\theta_i$. The other direction is analogous using the observation that $\{\alpha | (\mathfrak{A}_\alpha, q_\alpha) \models \vec{\exists}\theta_i\} \notin D$ if and only if $\{\alpha | (\mathfrak{A}_\alpha, q_\alpha) \models \vec{\exists}\theta_{i-1}\} \in D$ where $i \dot{-} 1 = 0$ if $i = 1$, $i \dot{-} 1 = 1$ if $i = 0$.

EXAMPLES. (a) Let Mod(Top) be the class of topological models, i.e. $(\mathfrak{A}, q)$ where q is a topology on A, and ULT(Top) the full topological ultraproduct defined in Sgro **[9]**. I(Mod(Top), ULT(Top)) is compact since

$$\mathrm{Seq}(\mathrm{Mod(Top)}) \times \mathrm{Ult} \subseteq \mathrm{Dom\ ULT(Top)}.$$

We also know that I(Mod(Top), ULT(Top)) has the Souslin-Kleene Property because ULT(Top) commutes with restriction. In fact, because $\mathfrak{L}^{\mathrm{top}}$ (confer Sgro **[11]**) is invariant and maximal with respect to a Łoś ultraproducts theorem, we have $I(\mathrm{Mod(Top)}, \mathrm{ULT(Top)}) = \mathfrak{L}^{\mathrm{top}}$.

(b) Let κ be a weakly compact cardinal, i.e. $\kappa \to (\kappa)^n_\gamma$ for all $\gamma < \kappa$ and $n \in \omega$. Take Mod(w.c.) to be the class of models, $(\mathfrak{A}, q)$, where q is the set of subsets of A of cardinality greater than or equal to κ. If D is a uniform ultrafilter on $\gamma < \kappa$, then $\mathrm{ULT(w.c.)}(\{(\mathfrak{A}_\alpha, q_\alpha)\}_{\alpha<\gamma}, D)$ is $(\Pi_D \mathfrak{A}_\alpha, q^*)$ where q^* is the collection of subsets of $\Pi_D A_\alpha$ of cardinality greater than or equal to κ. ($\Pi_D \mathfrak{A}_\alpha$ is the usual first-order ultraproduct.)

Because ULT(w.c.) commutes with restriction and $\mathrm{Seq\ Mod(w.c.)} \times \mathrm{Ult}^*(\gamma) \subseteq \mathrm{Dom\ ULT(w.c.)}$ for all $\gamma < \kappa$ we know that I(Mod(w.c.), ULT(w.c.)) is (ω, γ)-compact for $\gamma < \kappa$ and has the Souslin-Kleene Property.

Taking $\mathfrak{L}(Q_\kappa^{<\omega})$ to be the Malitz-Magidor quantifier under the κ interpretation, confer **[7]**, we claim that $\mathfrak{L}(Q_\kappa^{<\omega}) \lneq I(\mathrm{Mod(w.c.)}, \mathrm{ULT(w.c.)})$.

The inequality is straightforward because by **[6]** $\mathfrak{L}(Q_\kappa^{<\omega})$ does not have the Souslin-Kleene Property.

We need only show that each $Q^n_{x_1,\dots,x_n}\varphi(x_1,\dots,x_n)$ is equivalent to an invariant sentence. It is easy to see that the following formula of $\mathfrak{L}^2$ is equivalent to $Q^n_{x_1,\dots,x_n}\varphi(x_1,\dots,x_n)$,

$$\varphi^* = \exists X_1^q \forall x_1,\dots,x_n\Big(\bigvee\!\!\!\bigvee_{i\neq j} x_i = x_j \vee \bigvee\!\!\!\bigvee_i x_i \notin X_1^q \vee \bigwedge\!\!\!\bigwedge_{\sigma \text{ perm.}} \varphi(x_{\sigma(1)},\dots,x_{\sigma(n)})\Big).$$

We claim that it is invariant if $\varphi(x_1,\dots,x_n)$ is. If $\{\alpha|(\mathfrak{A}_\alpha, q_\alpha)\models\varphi^*\}\in D$ then $\mathrm{ULT(w.c.)}(\{(\mathfrak{A}_\alpha, q_\alpha)\}, D)\models\varphi^*$ is straightforward because existential sentences go up. So suppose that $\mathrm{ULT(w.c.)}(\{(\mathfrak{A}_\alpha, q_\alpha)\}, D)\models\varphi^*$. That is, there is a subset X of $\Pi_D A_\alpha$ of cardinality κ (which we will well-order by $<$) such that for each $a_1 < \cdots < a_n \in X$ we have $(\Pi_D\mathfrak{A}_\alpha, q^*)\models\bigwedge\!\!\!\bigwedge_\sigma \varphi(a_{\sigma(1)},\dots,a_{\sigma(n)})$. Hence by our assumption

$$\Big\{\alpha|(\mathfrak{A}_\alpha, q_\alpha)\models\bigwedge\!\!\!\bigwedge_\sigma \varphi(a_{\sigma(1)}(\alpha),\dots,a_{\sigma(n)}(\alpha))\Big\}\in D. \qquad (*)$$

Define $H\colon [X]^n\to D$ as follows: if $\{a_1<\cdots<a_n\}\in[X]^n$ then

$$H(\{a_1<\cdots<a_n\}) = \Big\{\alpha|(\mathfrak{A}_\alpha, q_\alpha)\models\bigwedge\!\!\!\bigwedge_\sigma \varphi(a_{\sigma(1)}(\alpha),\dots,a_{\sigma(n)}(\alpha))\wedge\bigwedge\!\!\!\bigwedge_{i\neq j} a_i(\alpha)\neq a_j(\alpha)\Big\}$$

which is in D by $(*)$.

Because κ is weakly compact and $|D|<\kappa$ we have that there is a $Y\subseteq X$ such that $H(Y)=\mathfrak{U}$ and $\mathfrak{U}\in D$.

Take $\alpha\in\mathfrak{U}$. Then we know that for every $a_1<\cdots<a_n\in Y$

$$(\mathfrak{A}_\alpha, q_\alpha)\models\bigwedge\!\!\!\bigwedge_\sigma \varphi(a_{\sigma(1)}(\alpha),\dots,a_{\sigma(n)}(\alpha))\wedge\bigwedge\!\!\!\bigwedge_{i\neq j} a_i(\alpha)\neq a_j(\alpha).$$

So $Y_\alpha=\{a(\alpha)|a\in Y\}$ has cardinality κ and

$$\mathfrak{U}\subseteq\Big\{\alpha|(\mathfrak{A}_\alpha, q_\alpha)\models\forall x_1,\dots,x_n\Big(\bigvee\!\!\!\bigvee_{i\neq j} x_i=x_j\vee\bigvee\!\!\!\bigvee_i x_i\notin Y_\alpha \vee\bigwedge\!\!\!\bigwedge_\sigma \varphi(x_{\sigma(1)},\dots,x_{\sigma(n)})\Big)\Big\}.$$

Hence $\{\alpha|(\mathfrak{A}_\alpha, q_\alpha)\models\varphi^*\}\in D$.

Question. Is $\Delta(\mathfrak{L}(Q_\kappa^{<\omega}))<I(\mathrm{Mod(w.c.)}, \mathrm{ULT(w.c.)})$? Confer **[6]**.

Question. What is the relationship between $\mathfrak{L}^{\mathrm{neg}}$ and I? Confer **[2]**.

(c) Let $\mathrm{Mod}(\omega_1)$ be the class of ω_1-standard models as in Keisler **[5]**, i.e., $(\mathfrak{A}, q)$ where q is the set of uncountable subsets of A. We will define an ultraproduct $\mathrm{ULT}(\omega_1)$ on $\mathrm{Mod}(\omega_1)$ such that $\mathfrak{L}(Q_{\omega_1})$ will be invariant. Confer **[5]**.

Let D be an ultrafilter on a countable set and $\{(\mathfrak{A}_i, q_i)\}_{i\in\omega}$ a sequence of ω_1-standard models. If D is principal then $\mathrm{ULT}(\{(\mathfrak{A}_i, q_i)\}_{i\in\omega}, D)=(\mathfrak{A}_j, q_j)$ where $\{j\}$ generates D.

If D is nonprincipal then it is ω_1-good, confer **[4]**. Hence $\Pi_D(\mathfrak{A}_i, q_i)$ (the standard many-sorted ultraprouct) is ω_1-saturated with respect to $\mathfrak{L}(Q_{\omega_1})$. Now by Keisler **[5]** there is an ω_1-standard model $(\mathfrak{B}, r)$, elementarily equivalent to $\Pi_D(\mathfrak{A}_i, q_i)$.

Thus if r^* is the $\mathfrak{L}(Q_{\omega_1})$-definable-over-$(\mathfrak{B}, r)$ subsets of B in r we have an $\mathfrak{L}(Q_{\omega_1})$-elementary embedding h of $(\mathfrak{B}, r^*)$ into $\Pi_D(\mathfrak{A}_i, q_i)$. Letting $h(r)$ be the uncountable subsets of $h(B)$ we have $(h(\mathfrak{B}), h(r)) \equiv_{\mathfrak{L}(Q_{\omega_1})} \Pi_D(\mathfrak{A}_i, q_i)$ and $h(\mathfrak{B}) \prec \Pi_D \mathfrak{A}_i$. Define $\mathrm{ULT}(\mathrm{Mod}(\omega_1), D) = (h(\mathfrak{B}), h(r))$. Then $\mathfrak{L}(Q_{\omega_1}) < I(\mathrm{Mod}(\omega_1), \mathrm{ULT}(\omega_1))$ which is countably compact.

Question. How strong is $I(\mathrm{Mod}(\omega_1), \mathrm{ULT}(\omega_1))$ for various choices of $(\mathfrak{B}, r)$?

(d) Assume $\diamond_{\omega_1}$, confer [7]. Using the completeness theorem proved in [7] for $\mathfrak{L}(Q_{\omega_1}^{<\omega})$ we define an ultraproduct on $\mathrm{Mod}(\omega_1)$ such that $\mathfrak{L}(Q_{\omega_1}^{<\omega})$ is invariant in an analogous fashion to (c). Also we can treat $\Delta(\mathfrak{L}(Q_{\omega_1}))$ and $\Delta(\mathfrak{L}(Q_{\omega_1}^{<\omega}))$ analogously.

REMARKS. We would like to note that in example (a) we could have taken any invariant logic (in the sense of [11]) and had the analogous result: i.e. $I(\mathrm{Mod}(\mathfrak{F}), \mathrm{ULT}(\mathfrak{F})) = \mathfrak{L}^{\mathfrak{F}}$. Also our choice of $\mathfrak{L}^2$ was somewhat arbitrary and we would like to point out that many other choices, e.g. logics with higher order variables would be equally good, see [8], which would have similar questions associated with them.

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