ULTRAPRODUCT INVARIANT LOGICS

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ABSTRACT. In this paper we give a construction of logics via the property of being preserved from the models to their ultraproduct. Specific examples are given which include some cardinality quantifiers.

We will assume that the reader is familiar with basic model theory including ultraproducts, Chang and Keisler [4], and abstract logics, confer Barwise [1].

DEFINITION. $(\mathcal{M}, \vec{q})$ is called a generalized model if $\vec{q}(i) \subseteq \mathcal{P}(A^{m(i)})$ for each $i$ (where $m(i) \in \omega$ and the function $m$ is called the type of the model).

Take $\text{Mod}$ to be a class of generalized models of the same type closed under restriction and expansion of languages.

Let $\text{Seq}_k(\text{Mod})$ be the class of all sequences of length $\kappa$ of elements of $\text{Mod}$, $\text{Ult}(\kappa)$ be the set of ultrafilters on $\kappa$, and $\text{Seq}(\text{Mod}) = \bigcup_k \text{Seq}_k(\text{Mod})$ and $\text{Ult} = \bigcup \text{Ult}(\kappa)$.

DEFINITION. An ultraproduct on $\text{Mod}$ is a partial function $\text{ULT}$ from $\bigcup_k \text{Seq}_k(\text{Mod}) \times \text{Ult}(\kappa)$ into $\text{Mod}$.

We say that the pair $\langle \text{Mod}, \text{ULT} \rangle$ is closed under $\kappa$-ultraproducts if $\text{Seq}^*(\kappa) \subseteq \text{Dom Ult}$. If $\langle \text{Mod}, \text{ULT} \rangle$ is closed under $\kappa$-ultraproducts for all cardinals $\kappa$ then it is said to be closed under ultraproducts.

Let $\langle \mathcal{L}_2, \mathcal{E}_2 \rangle$ be the second-order logic with additional quantification over the generalized domains. That is:

(a) the usual clauses for generalized first-order logics,

(b) if $t$ is a term and $X^\vec{q}(t)$ is a generalized variable then $t \in X^\vec{q}(t)$ is a formula and $(\mathcal{M}, \vec{q}) \models \varphi(t) \in X^\vec{q}(t)$ if and only if $s(i) \in s(X^\vec{q}(t))$ where $s(X^\vec{q}(t)) \in \vec{q}(i)$,

(c) if $R$ (or $f$) are formulas where $R$ (or $f$) is a relation (function) variable then $\exists Rp(R) (\exists fp(f))$ is a formula. $\exists Rp(R)$ and $\exists fp(f)$ have the usual second-order interpretation,

(d) if $\varphi(X^\vec{q}(t))$ is a formula then so is $\exists X^\vec{q}(t) \varphi(X^\vec{q}(t)). (\mathcal{M}, \vec{q}) \models \varepsilon \exists X^\vec{q}(t) \varphi(X^\vec{q}(t))$ if and only if there is an assignment $s$ such that $(\mathcal{M}, \vec{q}) \vDash \varphi(X^\vec{q}(t))[s]$.

Let $\text{Mod}$ and $\text{ULT}$ be given.

DEFINITION. A sentence $\varphi$ in the language $L$ of $\mathcal{L}_2$ is called invariant if and only if for each $\langle \langle \mathcal{M}_a, \vec{q}_a \rangle \rangle_{a \in \kappa}, D \rangle \in \text{Dom Ult}$ (each $\mathcal{M}_a, \vec{q}_a$ is an $L$-model) $a(\mathcal{M}_a, \vec{q}_a) \vDash \varphi \in \varphi$ if and only if $\text{ULT}(\langle \langle \mathcal{M}_a, \vec{q}_a \rangle \rangle_{a \in \kappa}, D) \vDash \varphi$.

Let $\text{I}(\text{Mod}), \text{ULT}^*$ be the sublogic of $\mathcal{L}_2$ consisting of the invariant sentences.

Let $\text{Ult}^*(\kappa)$ be the class of regular ultrafilters over $\kappa$. 

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THEOREM. If $\text{Seq}_{\kappa}(\text{Mod}) \times \text{Ult}^*{\kappa} \subseteq \text{Dom Ult}$ then $I(\text{Mod}, \text{ULT})$ is $(\omega, \kappa)$-compact. In fact it satisfies a $\kappa$ version of the L"os ultraproducts theorem.

PROOF. Straightforward by the definition of $I(\text{Mod}, \text{ULT})$.

THEOREM. Let $\text{ULT}$ commute with restriction. Then $I(\text{Mod}, \text{ULT})$ has the Souslin-Kleene Property (i.e. complementary $\text{PC}(L)$ classes are $\text{EC}(L)$ classes).

REMARK. For $\text{ULT}$ to commute with restriction we mean that if $\text{ULT}((\mathfrak{A}_a, q_a)), D) = (\exists \mathfrak{A}, r)$ then $(\exists \mathfrak{A} \uparrow L, r) = \text{ULT}((\mathfrak{A}_a \uparrow L, q_a)), D)).

PROOF. Let $\theta_0$ be a sentence of $L_0$, $\theta_1$ of $L_1$.

$$\Omega = \{ (\mathfrak{A} \uparrow L, \mathfrak{B}, q) \lvert (\mathfrak{A}, q) \models \theta_0 \},$$

$$\Omega^* = \{ (\mathfrak{A} \uparrow L, \mathfrak{B}, q) \lvert (\mathfrak{A}, q) \models \theta_1 \}.$$

We need to show that $\Omega$ is $\text{EC}(L_0 \cap L_1)$.

Define $\exists \theta_i$ to be the sentence obtained from $\theta_i$ by replacing all the relation and function symbols of $L_i - L_0 \cap L_1$ by variables and existentially quantifying over them.

We claim that $\exists \theta_i$ is an $L_0 \cap L_1$-invariant sentence. Suppose $\{ \alpha \lvert (\mathfrak{A}_a, q_a) \models \exists \theta_i \} \in D$. Then there are $\mathfrak{R}_a, f_a$ such that $\{ \alpha \lvert (\mathfrak{A}_a, \mathfrak{R}_a, f_a, q) \models \theta_i \} \in D$. Hence $\text{ULT}((\mathfrak{A}_a, \mathfrak{R}_a, f_a, q_a))_{a \in \kappa, D) \models \theta_i}$.

Using commutativity we get that $\text{ULT}((\mathfrak{A}_a, q_a)), D) \models \exists \theta_i$. The other direction is analogous using the observation that $\{ \alpha \lvert (\mathfrak{A}_a, q_a) \models \exists \theta_i \} \in D$ if and only if $\{ \alpha \lvert (\mathfrak{A}_a, q_a) \models \exists \theta_{i-1} \} \in D$ where $i - 1 = 0$ if $i = 1$, $i - 1 = 1$ if $i = 0$.

EXAMPLES. (a) Let $\text{Mod}(\text{Top})$ be the class of topological models, i.e. $(\mathfrak{A}, q)$ where $q$ is a topology on $A$, and $\text{ULT}(\text{Top})$ the full topological ultraproduct defined in Sgro [9]. $I(\text{Mod}(\text{Top}), \text{ULT}(\text{Top}))$ is compact since

$$\text{Seq}(\text{Mod}(\text{Top})) \times \text{Ult} \subseteq \text{Dom Ult}(\text{Top}).$$

We also know that $I(\text{Mod}(\text{Top}), \text{ULT}(\text{Top}))$ has the Souslin-Kleene Property because $\text{ULT}(\text{Top})$ commutes with restriction. In fact, because $\mathcal{L}^\text{Top}$ (confer Sgro [11]) is invariant and maximal with respect to a L"os ultraproducts theorem, we have $I(\text{Mod}(\text{Top}), \text{ULT}(\text{Top})) = \mathcal{L}^\text{Top}$.

(b) Let $\kappa$ be a weakly compact cardinal, i.e. $\kappa \rightarrow (\kappa)^n_\omega$ for all $\gamma < \kappa$ and $n \in \omega$. Take $\text{Mod}(w.c.)$ to be the class of models, $(\mathfrak{A}, q)$, where $q$ is the set of subsets of $A$ of cardinality greater than or equal to $\kappa$. If $D$ is a uniform ultrafilter on $\gamma < \kappa$, then $\text{ULT}(w.c.)((\mathfrak{A}_a, q_a))_{a < \gamma}, D) = (\Pi_D \mathfrak{A}_a, q^*)$ where $q^*$ is the collection of subsets of $\Pi_D A_a$ of cardinality greater than or equal to $\kappa$. ($\Pi_D \mathfrak{A}_a$ is the usual first-order ultraproduct.)

Because $\text{ULT}(w.c.)$ commutes with restriction and $\text{Seq} \text{Mod}(w.c.) \times \text{Ult}^*(\gamma) \subseteq \text{Dom Ult}(w.c.)$ for all $\gamma < \kappa$ we know that $I(\text{Mod}(w.c.), \text{ULT}(w.c.))$ is $(\omega, \gamma)$-compact for $\gamma < \kappa$ and has the Souslin-Kleene Property.

Taking $\mathcal{L}(Q_{<\omega}^{<\kappa})$ to be the Malitz-Magidor quantifier under the $\kappa$ interpretation, confer [7], we claim that $\mathcal{L}(Q_{<\omega}^{<\kappa}) \subseteq I(\text{Mod}(w.c.), \text{ULT}(w.c.))$.

The inequality is straightforward because by [6] $\mathcal{L}(Q_{<\omega}^{<\kappa})$ does not have the Souslin-Kleene Property.
We need only show that each $Q^n_{x_1,\ldots,x_n} \varphi(x_1,\ldots,x_n)$ is equivalent to an invariant sentence. It is easy to see that the following formula of $L^2$ is equivalent to $Q^n_{x_1,\ldots,x_n} \varphi(x_1,\ldots,x_n)$:

$$\varphi^* = \exists X \forall x_1,\ldots,x_n \left( \bigwedge_{i \neq j} x_i = x_j \lor \bigwedge_i x_i \notin X \lor \bigwedge_{i \neq \text{perm.}} \varphi(x_{\alpha(1)},\ldots,x_{\alpha(n)}) \right).$$

We claim that it is invariant if $\varphi(x_1,\ldots,x_n)$ is. If $\{a \mid (a, q_a) \in \varphi^*\} \in D$ then $\text{ULT}(w.c.)(((\mathfrak{A}_a, q_a)), D) \models \varphi^*$ is straightforward because existential sentences go up. So suppose that $\text{ULT}(w.c.)(((\mathfrak{A}_a, q_a)), D) \models \varphi^*$. That is, there is a subset $X$ of $\Pi_D \mathfrak{A}_a$ of cardinality $\kappa$ (which we will well-order by $<$) such that for each $a_1 < \cdots < a_n \in X$ we have $(\Pi_D \mathfrak{A}_a, q^*) \models \bigwedge_{\alpha \in P} \varphi(x_{\alpha(1)},\ldots,x_{\alpha(n)})$. Hence by our assumption

$$\bigl\{ a \mid (a, q_a) \models \bigwedge_{\alpha} \varphi(a_{\alpha(1)}(\alpha),\ldots,a_{\alpha(n)}(\alpha)) \bigr\} \in D.$$  

Define $H : [X]^n \to D$ as follows: if $\{a_1 < \cdots < a_n\} \subseteq [X]^n$ then

$$H(\{a_1 < \cdots < a_n\}) = \bigl\{ a \mid (a, q_a) \models \bigwedge_{\alpha} \varphi(a_{\alpha(1)}(\alpha),\ldots,a_{\alpha(n)}(\alpha)) \land \bigwedge_{i \neq j} a_i(\alpha) \neq a_j(\alpha) \bigr\}$$

which is in $D$ by ($\ast$).

Because $\kappa$ is weakly compact and $|D| < \kappa$ we have that there is a $Y \subseteq X$ such that $H(Y) = \mathfrak{U}$ and $\mathfrak{U} \in D$.

Take $a \in \mathfrak{U}$. Then we know that for every $a_1 < \cdots < a_n \in Y$

$$(a, q_a) \models \bigwedge_{\alpha} \varphi(a_{\alpha(1)}(\alpha),\ldots,a_{\alpha(n)}(\alpha)) \land \bigwedge_{i \neq j} a_i(\alpha) \neq a_j(\alpha).$$

So $Y_a = \{a(\alpha) \mid a \in Y\}$ has cardinality $\kappa$ and

$$\mathfrak{U} \subseteq \bigl\{ a \mid (a, q_a) \models \forall x_1,\ldots,x_n \left( \bigwedge_{i \neq j} x_i = x_j \lor \bigwedge_i x_i \notin Y_a \lor \bigwedge_{i \neq \text{perm.}} \varphi(x_{\alpha(1)},\ldots,x_{\alpha(n)}) \right) \bigr\}.$$  

Hence $\{a \mid (a, q_a) \models \varphi^*\} \in D$.

**Question.** Is $\Delta (\mathcal{L}(\mathcal{Q}_\omega^{<\omega})) < I(\text{Mod}(w.c.), \text{ULT}(w.c.))$? Confer [6].

**Question.** What is the relationship between $\mathcal{L}_{\text{neg}}$ and $I$? Confer [2].

(c) Let $\text{Mod}(\omega_1)$ be the class of $\omega_1$-standard models as in Keisler [5], i.e., $(\mathfrak{A}, q)$ where $q$ is the set of uncountable subsets of $A$. We will define an ultraproduct $\text{ULT}(\omega_1)$ on $\text{Mod}(\omega_1)$ such that $\mathcal{L}(\mathcal{Q}_{\omega_1})$ will be invariant. Confer [5].

Let $D$ be an ultrafilter on a countable set and $\{((\mathfrak{A}_i, q_i))_{i \in \omega}\}$ a sequence of $\omega_1$-standard models. If $D$ is principal then $\text{ULT}(((\mathfrak{A}_i, q_i))_{i \in \omega}, D) = ((\mathfrak{A}_i, q_i))$ where $\{j\}$ generates $D$.

If $D$ is nonprincipal then it is $\omega_1$-good, confer [4]. Hence $\Pi_D (\mathfrak{A}_i, q_i)$ (the standard many-sorted ultraproduct) is $\omega_1$-saturated with respect to $\mathcal{L}(\mathcal{Q}_{\omega_1})$. Now by Keisler [5] there is an $\omega_1$-standard model $((\mathfrak{A}, r), r)$, elementarily equivalent to $\Pi_D (\mathfrak{A}_i, q_i)$.

Thus if $r^*$ is the $\mathcal{L}(\mathcal{Q}_{\omega_1})$-definable-over-$(\mathfrak{A}, r)$ subsets of $B$ in $r$ we have an $\mathcal{L}(\mathcal{Q}_{\omega_1})$-elementary embedding $h$ of $(\mathfrak{A}, r^*)$ into $\Pi_D (\mathfrak{A}_i, q_i)$. Letting $h(r)$ be the
uncountable subsets of \( h(B) \) we have \( (h(\mathcal{B}), h(r)) \equiv_{\ell(Q_{\omega_1})} \Pi_\mathcal{D}(\mathcal{F}, q_i) \) and \( h(\mathcal{B}) < \Pi_\mathcal{D}(\mathcal{F}, q_i) \). Define \( \text{ULT}(\text{Mod}(\omega_1), D) = (h(\mathcal{B}), h(r)) \). Then \( \ell(Q_{\omega_1}) < I(\text{Mod}(\omega_1), \text{ULT}(\omega_1)) \) which is countably compact.

**Question.** How strong is \( I(\text{Mod}(\omega_1), \text{ULT}(\omega_1)) \) for various choices of \((\mathcal{B}, r)\)?

(d) Assume \( \mathcal{B} \), confer [7]. Using the completeness theorem proved in [7] for \( \ell(Q_{\omega_1}) \) we define an ultraproduct on \( \text{Mod}(\omega_1) \) such that \( \ell(Q_{\omega_1}) \) is invariant in an analogous fashion to (c). Also we can treat \( \Delta(\ell(Q_{\omega_1})) \) and \( \Delta(\ell(Q_{\omega_1})) \) analogously.

**Remarks.** We would like to note that in example (a) we could have taken any invariant logic (in the sense of [11]) and had the analogous result: i.e. \( I(\text{Mod}(\mathcal{B}), \text{ULT}(\mathcal{B})) = \ell^\mathcal{B} \). Also our choice of \( \mathcal{B} \) was somewhat arbitrary and we would like to point out that many other choices, e.g. logics with higher order variables would be equally good, see [8], which would have similar questions associated with them.

**References**


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