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ON BIEBERBACH'S ANALYSIS OF DISCRETE EUCLIDEAN GROUPS

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ABSTRACT. For a subgroup G of the euclidean group $E_n = O_n \cdot \mathbf{R}^n$ (semidirect product) and a real number $r > 0$, let G^* denote the translation subgroup of G, G_r the group generated by all (A, a) in G with $\|1 - A\| < r$ (operator norm), and $k_n(r)$ the maximum number of elements of O_n with mutual distances $> r$ relative to the metric $d(A, B) = \|A - B\|$. We give an elementary, largely geometrical proof of the following results of Bieberbach: Let G be a subgroup of E_n. (1) If G is discrete, then $G_{1/2}$ is abelian, $G_{1/2} \lhd G$, and $[G: G_{1/2}] < k_n(1/2)$. (2) G is discrete if and only if $G \subset O_{n-k} \times E_k$, where p_2G is discrete, $(p_2G)^*$ spans $\mathbf{R}^k$, and $G \cap \ker p_2$ is finite. (Here p_2 is the projection on the second factor.) (3) G is crystallographic if and only if G is discrete and G^* spans $\mathbf{R}^n$. Moreover, if G is crystallographic, then $[G: G^*] < k_n(1/2)$.

The purpose of this paper is to give a proof of Bieberbach's main results on discrete euclidean groups, which is no more complicated than any of the existing proofs, and which bridges the gap between the 3-dimensional case and the n-dimensional case. These results are as follows (where by a point group we mean a euclidean group whose elements have a common invariant point and by a space group a euclidean group whose translations span the underlying space):

THEOREM 1. *A discrete euclidean group has an abelian normal subgroup containing the translation subgroup with index bounded by a number depending only on the dimension of the underlying space.*

THEOREM 2. *A euclidean group is discrete if and only if it is a subdirect product of a point group and a discrete space group such that the group of all elements of the point group paired with the identity element of the space group is finite.*

THEOREM 3. *A euclidean group is crystallographic if and only if it is a discrete space group. Moreover, if G is crystallographic, then the index in G of the translation subgroup of G is bounded by a number depending only on the dimension of the underlying space.*

There are essentially five proofs of the first assertion of Theorem 3, which is apparently the most important of the above results. These are due to Bieberbach [3], Frobenius [6], Zassenhaus [13], Auslander [1], and Vinberg [11]. The proof of Bieberbach is unnecessarily complicated and is almost entirely superseded by the

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proof of Frobenius. The proof of Zassenhaus is not really comparable with the others, since Zassenhaus' main object is to establish a substantial generalization of Theorem 1, while that of the other authors is just to obtain a proof of the first assertion of Theorem 3 which is as simple as possible. The proofs of Auslander and Vinberg are probably best regarded as alternatives to the proof of Frobenius which provide additional insight. They definitely do not supersede it, since the proof of Frobenius yields proofs of Theorems 1 and 2 while those of Auslander and Vinberg do not, and also a proof of the second assertion of Theorem 3 which is simpler than the one of Auslander.

The present proof is essentially that of Frobenius. Thus we first prove Theorem 1 by making two simple additions to Bieberbach's proof [**2**] of the corresponding result of Jordan, and then obtain Theorems 2 and 3 as easy corollaries. These additions are Lemma 4, which corresponds to the case of parallel axes in the 3-dimensional case, and the second half of the proof of Lemma 5, which is also the direct analogue of a 3-dimensional argument.

However, the present proof is considerably simpler than any of the existing versions of the proof of Frobenius ([**6**], [**8**, pp. 382–393], [**7**, pp. 214–221]) and not really more complicated than the proof of Auslander or the proof of Vinberg. The simplifications are mainly the result of replacing analytical arguments by geometrical arguments. Two obvious examples of this replacement are the proofs of Lemmas 1 and 4. Another example is the second half of the proof of Lemma 5. Here the simple geometrical device of the ball K makes it possible to avoid the fairly complicated calculation and estimate of the translative part of a commutator. Moreover, this device provides additional insight. For if the scale is chosen so that the ball K looks like a point, then the action of any term of the sequence of transforms (or the corresponding sequence of commutators) is indistinguishable from the action of its orthogonal part. Thus, in a sense, the proof of Lemma 5 is reduced to the main part of Bieberbach's proof of Jordan's theorem. A notable simplification of a somewhat different type is the proof of Theorem 2. This is clearly much simpler than the proof of Bieberbach [**3**, pp. 334–336], and it is also, I claim, an improvement on the proof of Rinow [**8**, pp. 389–393, p. 394 (last paragraph)], which is apparently the only other existing proof.

Finally, of all the existing proofs of the first assertion of Theorem 3, the present proof is the only one that is readily modified to give a convenient geometrical proof for the 3-dimensional case, and thus, in a sense, the only one to bridge the gap between the 3-dimensional case and the n-dimensional case. The necessary modification is indicated in Remark 2 at the end of the paper, and the corresponding proof is no more complicated than the recent one of Delone and Štogrin [**5**].

NOTATION. E_n, O_n, and T_n will denote respectively the euclidean, orthogonal, and translation groups of $\mathbf{R}^n$. E_n will be identified with the semidirect product $O_n \cdot \mathbf{R}^n$, i.e. the product set $O_n \times \mathbf{R}^n$ with the multiplication $(B, b)(A, a) = (BA, Ba + b)$. For a subgroup G of E_n, G^* and G^+ will denote respectively the translation and sense-preserving subgroups of G. For $x, y \in \mathbf{C}^n$, (x, y) will denote the standard inner product $\Sigma x_i \bar{y}_i$. And, for $A \in \mathrm{End}(\mathbf{R}^n)$, $\|A\|$ will denote the operator norm of A, i.e. $\|A\| = \sup\{|Ax|: |x| = 1\}$.

LEMMA 1. *Let $A, B \in O_n$. If $\|1 - B\| < \sqrt{2}$ and BAB^{-1} commutes with A, then B commutes with A.*

PROOF. (Cf. [**4**].) The condition $\|1 - B\| < \sqrt{2}$ is equivalent to the condition that B is a rotation (i.e. sense-preserving) with all angles $< \pi/2$, as is clear from the real normal form. Hence the eigenvalues of B have positive real parts. Hence $\operatorname{Re}(Bx, x) > 0$ for $0 \neq x \in \mathbf{C}^n$, and so B cannot send any nonzero vector of $\mathbf{C}^n$ to an orthogonal vector. Now let $V_1, \ldots, V_k$ be the eigenspaces of A. Then clearly $BV_1, \ldots, BV_k$ are the eigenspaces of BAB^{-1}. Thus, since BAB^{-1} commutes with A, we have $BV_i = \bigoplus_j (BV_i \cap V_j)$. Now, since B cannot send any nonzero vector of $\mathbf{C}^n$ to an orthogonal vector, we have $BV_i \cap V_j = \{0\}$ for $j \neq i$. Thus $BV_i = BV_i \cap V_i \subset V_i$, and so B commutes with A.

LEMMA 2. *If $A, B \in O_n$, then $\|BAB^{-1} - A\| \leqslant 2\|1 - A\|\,\|B - A\|$.*

PROOF. This follows from the identity

$$BAB^{-1} - A = ((1 - A)(B - A) - (B - A)(1 - A))B^{-1}$$

and the properties of the operator norm.

LEMMA 3. *Let C be a conjugacy class of E_n with $\|1 - A\| < 1$ for $(A, a) \in C$. Then any sufficiently large ball $K \subset \mathbf{R}^n$ has the following property: Any element of C whose axis meets K maps the center of K to a point of K.*

PROOF. (Cf. [**10**].) Let m denote the common value of $\|1 - A\|$ for $(A, a) \in C$, and let t denote the common translation-component length of the elements of C. And suppose that f is an element of C with axis tangent to a ball K of center x and radius r. Now apply the rotative part of f to x, obtaining y, and then the translative part of f to y, obtaining z. Then clearly the distance from x to y is $\leqslant mr$ and that from y to z is t. Hence, since the segments xy and yz are perpendicular, the distance d from x to z satisfies $d^2 \leqslant m^2r^2 + t^2$. Thus, since $m < 1$, it is clear that for r sufficiently large we have $d^2 \leqslant r^2$ and therefore $f(x) = z \in K$. Now to complete the proof we simply note that if we move the axis of f toward x we have $f(x) \in K$ with more room to spare.

LEMMA 4. *Let G be a discrete subgroup of E_n. Suppose $g = (A, a)$, $h = (B, b) \in G$ and $\|1 - A\| < 1$, $\|1 - B\| < 1$ and $BA = AB$. Then $hg = gh$.*

PROOF. Since a pair of commuting orthogonal matrices can be simultaneously transformed to normal form, we have $g, h \in (E_2^+)^p \times (E_1^+)^q$ $(2p + q = n)$. Now suppose if possible that $hg \neq gh$. Then for some i, $1 \leqslant i \leqslant p$, the ith components g^i and h^i do not commute, and by interchanging g and h we may assume that g^i is a rotation. Now define $g_1 = hgh^{-1}$, $g_2 = g_1 g g_1^{-1}$, $g_3 = g_2 g g_2^{-1}$, etc. Then, since $\|1 - A\| < 1$ and therefore the angle of g^i is $< \pi/3$, the centers of the g_k^i spiral in logarithmically to the center of g^i (Figure 1). Hence the g_k^i and therefore the g_k are distinct, and the g_k^i converge to g^i. Further, the g_k^j converge to g^j for $j \neq i$. For if g^j

and h^j commute we have $g_k^j = g^j$ for $k \geqslant 1$, and if they do not commute we are either in the above situation or in the situation where g^j is a translation and h^j is a rotation, and here (as one easily checks geometrically) we have $g_k^j = g^j$ for $k \geqslant 2$. Thus the g_k are distinct and converge to g, contrary to the discreteness of G. Therefore we must have $hg = gh$.

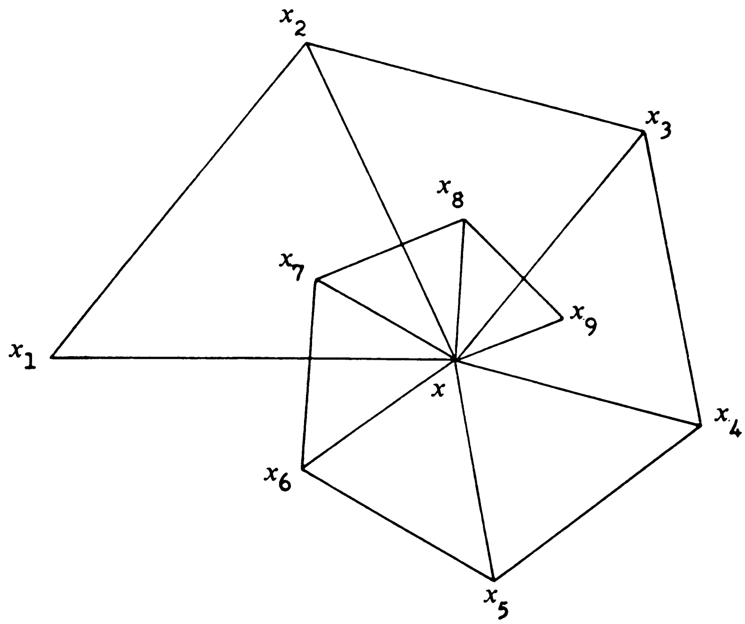

FIGURE 1
(x = center of g^i, x_k = center of g_k^i)

LEMMA 5. *Let G be a discrete subgroup of E_n. Suppose $g = (A, a)$, $h = (B, b) \in G$ and $\|1 - A\| < 1/2$, $\|1 - B\| < 1/2$. Then $BA = AB$.*

PROOF. Suppose if possible that $BA \neq AB$. And let $g_k = (A_k, a_k)$ be the sequence of transforms $g_1 = hgh^{-1}$, $g_2 = g_1 g g_1^{-1}$, $g_3 = g_2 g g_2^{-1}$, etc. Then $A_k = A_{k-1} A A_{k-1}^{-1}$ for $k \geqslant 1$ (where $A_0 = B$). Hence, since $\|1 - A_k\| = \|A_{k-1}(1 - A)A_{k-1}^{-1}\| = \|1 - A\| < \sqrt{2}$ for $k \geqslant 1$, it follows from Lemma 1 by induction that A_k never commutes with A and therefore that A_k never equals A. Hence, by Lemma 2, A_{k+1} is strictly closer to A than A_k is. Thus the A_k and therefore the g_k are distinct. Now let (by Lemma 3) K be a ball about a point x on the axis of g with the following two properties: (1) Any transform (= conjugate) of g whose axis meets K maps x to a point of K. (2) The axis of g_1 meets K. Then by induction the axis of every g_k meets K. Hence, taking x to be the origin, we have $a_k = g_k(0) \in K$, and so the g_k are bounded (the A_k being automatically bounded). But since the g_k are distinct this is contrary to the discreteness of G. Therefore we must have $BA = AB$.

LEMMA 6. *Let G be any subgroup of E_n and r a real number > 0. Let G_r denote the subgroup of G generated by all elements (A, a) in G with $\|1 - A\| < r$. Let $k_n(r)$ denote the maximum number of elements of O_n with mutual distances $\geqslant r$ relative to the metric $d(A, B) = \|A - B\|$. Then $G_r \lhd G$ and $[G: G_r] \leqslant k_n(r)$.*

PROOF. Both assertions follow easily from the orthogonal invariance of the operator norm.

PROOF OF THEOREM 1. Let G be a discrete subgroup of E_n. Then, by Lemmas 4, 5, and 6, $G_{1/2}$ is a subgroup of the required kind.

PROOF OF THEOREM 2. Let G be a discrete subgroup of E_n and put $H = G_{1/2}$. Then, since H is abelian, we have $H \subset O_i \times T_j \times T_k$ $(i + j + k = n)$, where p_1H leaves invariant only the origin, p_2H consists of just the identity, and p_3H spans $\mathbf{R}^k$. (Henceforth p_i denotes the projection on the ith factor of the direct product in question.) Thus, since H is normal in G, we have $G \subset O_i \times E_j \times E_k$. Now, since H has finite index in G, it follows that p_2G is finite and therefore a point group, for clearly $|p_2G| = [p_2G: p_2H] \leqslant [G: H]$. Thus, after a change of origin, we have $G \subset O_{i+j} \times E_k$, where $(p_2G)^*$ spans $\mathbf{R}^k$ (a new p_2). Thus G is a subdirect product of a point group and a space group. The proofs of the remaining assertions are straightforward and easy and are therefore omitted.

PROOF OF THEOREM 3. Let G be a crystallographic subgroup of E_n. Then, since G is discrete, $G_{1/2}$ is abelian. And, since G is uniform and $G_{1/2}$ has finite index in G, $G_{1/2}$ is uniform. Thus, as is clear from the normal form for an abelian euclidean group, $G_{1/2}$ must consist entirely of translations which span $\mathbf{R}^n$. Thus G^* spans $\mathbf{R}^n$, i.e. G is a space group, and $[G: G^*]$ is bounded by the number $k_n(1/2)$ depending only on n. Since the sufficiency is clear, this completes the proof. (Note that the first assertion of this theorem also follows immediately from Theorem 2, and that the second assertion also follows from the proof of Theorem 2.)

REMARK 1. (Cf. [7, p. 216], [**12**, pp. 101–102].) In the proof of Lemma 5, one may show that in fact the g_k converge to g, as follows. Put $a_k = x_k + y_k$ where $A_kx_k = x_k$ and y_k is perpendicular to the axis ($= +1$ eigenspace) of A_k. Then, taking the origin to be on the axis of g so that $Aa = a$, we have $x_{k+1} = A_ka$ and $y_{k+1} = (1 - A_{k+1})a_k$. (To see this, first transform (A, a) by $(A_k, 0)$ and then transform the result $(A_kAA_k^{-1}, A_ka)$ by $(1, a_k)$.) Further, put $u_k = \|A_k - A\|$, $t = |x_k|$, $r_k = |y_k|$, $m = \|1 - A\|$. Then by Lemma 2 we have $u_{k+1} \leqslant 2mu_k$. Hence $A_k \to A$ and $x_k \to a$. Now, since $A_kx_k = x_k$, we have $y_{k+1} = (A_k - A_{k+1})x_k + (1 - A_{k+1})y_k$. Thus, since $\|A_k - A_{k+1}\| = \|A_k(A_k - A)A_k^{-1}\| = \|A_k - A\|$, we have $r_{k+1} \leqslant u_kt + mr_k$. Hence by induction $r_{k+1} \leqslant (2^k - 1)m^{k-1}tu_1 + m^kr_1$ and so $r_k \to 0$. Alternative argument assuming only that $m < 1$ and $u_k \to 0$: Fix a number n such that $m < n < 1$. Let v be such that $u_kt + mv \leqslant v$ $(k \geqslant 1)$ and $r_1 \leqslant v$. Then, if $r_k \leqslant v$, we have $r_{k+1} \leqslant u_kt + mr_k \leqslant v$. Hence, $r_k \leqslant v$ $(k \geqslant 1)$. Now choose $K_1 > 1$ so that $u_{k-1}t \leqslant (n - m)v$ $(k \geqslant K_1)$. Then $r_k \leqslant u_{k-1}t + mr_{k-1} \leqslant (n - m)v + mv = nv$ $(k \geqslant K_1)$. Similarly if $K_2 > K_1$ is such that $u_{k-1}t \leqslant (n - m)(nv)$ $(k \geqslant K_2)$, then $r_k \leqslant n(nv) = n^2v$ $(k \geqslant K_2)$, etc. Hence $r_k \to 0$.

REMARK 2. (Cf. [9], [10].) In the 3-dimensional case one may establish Lemma 5 under the weaker hypothesis that $\|1 - A\| < 1$, $\|1 - B\| < 1$, i.e. that A and B are rotations with angles $< \pi/3$, as follows. It is enough to show that the A_k in the first half of the proof are distinct. Let s_k denote the sine of the angle between the axis of A_k and that of A and put $m = \|1 - A\|$. Then, noting that A_k rotates the axis of A into that of A_{k+1}, we easily find that $s_{k+1} \leqslant ms_k$. Thus (roughly speaking) the axes of the A_k form a spiral analogous to that of Figure 1, and so the A_k are distinct. Further, one may show under this weaker hypothesis that the g_k converge to g, as follows. Let x_k, y_k, t, r_k be as in Remark 1 and suppose that $Aa = a$. Then since $s_{k+1} \leqslant ms_k$ it follows that $A_k \to A$ and $x_k \to a$. Now let x_k' be the projection of x_k on the plane through the origin perpendicular to the axis of A_{k+1}. Then, since the axis of A_k makes the same angle with the axes of A and A_{k+1}, we have $|x_k'| = ts_k$. Thus, since $y_{k+1} = (1 - A_{k+1})x_k' + (1 - A_{k+1})y_k$, we have $r_{k+1} \leqslant mts_k + mr_k$. Hence by induction $r_{k+1} \leqslant km^k ts_1 + m^k r_1$ and so $r_k \to 0$.

REMARK 3. If G is a crystallographic subgroup of E_n, then we have in fact $[G\colon G^*] \leqslant k_n(1)$. Proof: Let $g = (A, a)$ be an element of G with $\|1 - A\| < 1$. Transform g by a translation of G whose projection on one of the characteristic invariant planes of A is $\neq 0$, obtaining h, and define $g_1 = hgh^{-1}$, $g_2 = g_1 g g_1^{-1}$, $g_3 = g_2 g g_2^{-1}$, etc. Then, by considering the components of the g_k corresponding to the characteristic invariant planes of A, we see immediately that the g_k are distinct and converge to g. But this is contrary to the discreteness of G. Hence A has no characteristic invariant planes, i.e. $A = 1$. Hence $G_1 = G^*$, which implies the desired result.

NOTE ADDED IN PROOF. In the recent book *Crystallographic groups of four-dimensional space* by H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, and H. Zassenhaus, there is a new method for proving Theorems 2 and 3 quite close to that of Vinberg, which came to my attention after the present paper was written. However, this method does not yield a proof of Theorem 1 and it does not yield a convenient geometrical proof for the 3-dimensional case.

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