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A CLASS OF FINITE GROUP-AMALGAMS

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ABSTRACT. Let A_{-1} and A_1 be finite groups such that $A_{-1} \cap A_1 = A_0$ is a common subgroup with $[A_{-1} : A_0] = 4$ and $[A_1 : A_0] = 2$. We further assume that only the trivial subgroup of A_0 is normal in both A_{-1} and A_1. Let K be the intersection of all conjugates xA_0x^{-1} for $x \in A_{-1}$. Then if $A_0 \neq \{1\}$ we have $A_{-1}/K \cong D_4$, A_4, or S_4. We describe in detail all such amalgams (A_{-1}, A_1) when $A_{-1}/K \cong D_4$ (dihedral group of order 8). There are infinitely many of them, while if $A_{-1}/K \cong A_4$ or S_4 there are only finitely many amalgams.

1. An *amalgam* A is an ordered 5-tuple $A = (A_{-1}, \alpha_{-1}, A_0, \alpha_1, A_1)$ where A_{-1}, A_0, A_1 are groups and α_i: $A_0 \to A_i$ $(i = \pm 1)$ are monomorphisms. This amalgam is *trivial* if A_{-1}, A_0, and A_1 are trivial groups.

Let $B = (B_{-1}, \beta_{-1}, B_0, \beta_1, B_1)$ also be an amalgam. A *morphism* f: $A \to B$ is an ordered triple $f = (f_{-1}, f_0, f_1)$ of homomorphisms f_i: $A_i \to B_i$ such that $f_i \circ \alpha_i = \beta_i \circ f_0$ $(i = \pm 1)$ and $f_i(A_i) \cap \beta_i(B_0) = \beta_i f_0(A_0)$ $(i = \pm 1)$.

We shall say that A is a *subamalgam* of B (we then write $A \leqslant B$) if $A_i \leqslant B_i$ for $i = 0, \pm 1$ and the inclusion maps f_i: $A_i \to B_i$ give a morphism $f = (f_{-1}, f_0, f_1)$. It follows from this definition that if $A \leqslant B$ then $A_i \cap \beta_i(B_0) = \alpha_i(A_0)$ holds for $i = \pm 1$. A subamalgam A of B is *proper* if $A_{-1} < B_{-1}$ or $A_1 < B_1$.

A subamalgam $A \leqslant B$ is *normal* (we then write $A \lhd B$) if $A_i \lhd B_i$ for $i = 0, \pm 1$. If $A \lhd B$ then we define the *quotient amalgam* B/A by

$$B/A = (B_{-1}/A_{-1}, \gamma_{-1}, B_0/A_0, \gamma_1, B_1/A_1)$$

where γ_i: $B_0/A_0 \to B_i/A_i$ is the map induced by the composite of β_i: $B_0 \to B_i$ and the canonical map $B_i \to B_i/A_i$. It is easy to check that the canonical maps $B_i \to B_i/A_i$ $(i = 0, \pm 1)$ give a morphism $B \to B/A$.

An amalgam A is *simple* if A has no proper nontrivial normal subamalgams. We say that an amalgam A is *faithful* if $N \leqslant A_0$ and $\alpha_i(N) \lhd A_i$ $(i = \pm 1)$ imply that N is trivial. The *degree* d of an amalgam A is the ordered pair $d = (d_{-1}, d_1)$ where $d_i = [A_i : \alpha_i(A_0)]$.

Finally an amalgam A is *finite* if A_i $(i = 0, \pm 1)$ are finite groups.

2. Finite faithful amalgams of degree (3, 2) were studied in [**1**]. There are precisely seven of them (up to isomorphism). Finite faithful amalgams of degree (3, 3) were determined by D. M. Goldschmidt [**6**]. There are fifteen of them (up to isomorphism) provided that one considers the amalgams $(A_{-1}, \alpha_{-1}, A_0, \alpha_1, A_1)$ and $(A_1, \alpha_1, A_0, \alpha_{-1}, A_{-1})$ as the same.

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In this paper we consider only finite faithful amalgams A of degree (4, 2). It will be convenient to consider α_{-1} and α_1 as inclusions. Thus we have $A_{-1} \cap A_1 = A_0$. Assume that A_0 is not the trivial group and let

$$K = \bigcap_{x \in A_{-1}} x A_0 x^{-1}.$$

Since $[A_1 : A_0] = 2$ we have $A_0 \lhd A_1$. Since A is faithful and $A_0 \neq \{1\}$, we have $K < A_0$. The group A_{-1}/K acts faithfully on A_{-1}/A_0 and so it is isomorphic to one of the groups

D_4 (dihedral group of order 8),

A_4 (alternating group of order 12),

S_4 (the symmetric group of order 24).

If $A_{-1}/K \cong D_4$ we shall say that A is of *dihedral type*. There are infinitely many nonisomorphic finite faithful amalgams of degree (4, 2) of dihedral type. In this paper we give explicit presentations for all such amalgams. One can easily deduce from some results of A. Gardiner [3] and [4] that there are only finitely many nonisomorphic finite faithful amalgams A of degree (4, 2) with $A_{-1}/K \cong A_4$ or S_4.

It should also be mentioned that in view of a result of C. Sims [8] only a few of our amalgams (in all of them $|K| = 1, 2$, or 4) arise from finite primitive permutation groups whose 1-point stabilizer has an orbit of length four.

We shall write $x \leftrightarrow y$ instead of $xy = yx$. We write $Z(G)$ for the center of a group G, and G' for its derived group. Our notation for commutators is $[x, y] = xyx^{-1}y^{-1}$.

For connections between finite faithful amalgams of degree $(d, 2)$ and 1-transitive actions of groups on d-valent connected graphs we refer the reader to [2].

3. We can now state our result.

THEOREM. *Finite faithful amalgams of degree* (4, 2) *of dihedral type are* (*up to isomorphism*) *precisely the amalgams* $A = (A_{-1}, \alpha_{-1}, A_0, \alpha_1, A_1)$ *which have the presentation described below.*

Let n (≥ 2) *be an integer and let* m *be the smallest integer such that* $3m > 2n$. *We define the group* N *by the presentation*:

$$N = \langle a_0, a_1, \ldots, a_{n-1} \rangle,$$

$$a_i^2 = 1, \qquad 0 \leq i \leq n-1; \tag{1}$$

$$a_i \leftrightarrow a_j \quad \textit{for } 0 < j - i < m; \tag{2}$$

$$[a_i, a_j] = a_{n-m+i}^{\varepsilon(j-i,0)} a_{n-m+i+1}^{\varepsilon(j-i,1)} \cdots a_{m+j-n}^{\varepsilon(j-i,j-i+2m-2n)} \tag{3}$$

for $j - i \geq m$, *where* $\varepsilon(r, s) = 0$ *or* 1 *satisfy the symmetry condition*

$$\varepsilon(r, s) = \varepsilon(r, r - s + 2m - 2n) \tag{4}$$

for all r, s ($m \leq r \leq n-1$, $0 \leq s \leq r + 2m - 2n$).

Then we take $A_0 \leqslant N$ where A_0 is generated by $a_1, a_2, \ldots, a_{n-1}$. Furthermore

$$A_{-1} = \langle x, N \rangle, \qquad A_1 = \langle A_0, y \rangle,$$

$$x a_i x^{-1} = a_{n-1-i}, \qquad 0 \leqslant i \leqslant n-1; \tag{5}$$

$$y a_i y^{-1} = a_{n-i}, \qquad 1 \leqslant i \leqslant n-1; \tag{6}$$

$$\text{if } n \text{ odd} \quad y^2 = 1, \quad x^2 = 1 \text{ or } a_{(n-1)/2}; \tag{7'}$$

$$\text{if } n \text{ even} \quad x^2 = 1, \quad y^2 = 1 \text{ or } a_{n/2}. \tag{7''}$$

The maps α_{-1} and α_1 are inclusions and we have $|A_0| = 2^{n-1}$.

PROOF. Assume first that A is an amalgam which is given by the presentation in the theorem. Then

$$N' \leqslant \langle a_{n-m}, a_{n-m+1}, \ldots, a_{m-1} \rangle \leqslant Z(N);$$

in particular, N is nilpotent of class $\leqslant 2$. It follows that $X = \langle a_0, \ldots, a_{m-1} \rangle$ and $Y = \langle a_m, a_{m+1}, \ldots, a_{n-1} \rangle$ are elementary abelian groups of orders 2^m and 2^{n-m}, respectively, and that N is a semidirect product $N = X \rtimes Y$. The action of Y on X is described by the relations (3).

The relations (1)–(3) being invariant under the shifting of subscripts, it follows that there is an isomorphism

$$\sigma: \langle a_0, a_1, \ldots, a_{n-2} \rangle \to A_0 = \langle a_1, a_2, \ldots, a_{n-1} \rangle$$

such that $\sigma(a_i) = a_{i+1}$, $0 \leqslant i \leqslant n-2$. Because of (4) there is an automorphism θ of N such that $\theta(a_i) = a_{n-1-i}$, $0 \leqslant i \leqslant n-1$. Consequently $|A_{-1}| = 2|N|$.

Since $(\sigma \circ \theta)(a_i) = \sigma(a_{n-1-i}) = a_{n-i}$ for $1 \leqslant i \leqslant n-1$, it follows that $|A_1| = 2|A_0|$.

Thus A is indeed an amalgam of degree (4, 2) and $|A_0| = 2^{n-1} \geqslant 2$. The relations (5) and (6) imply that A is faithful, and clearly it is of dihedral type.

Now assume that A is a finite faithful amalgam of degree (4, 2) of dihedral type. Then A_0 is necessarily a nontrivial 2-group. Thus $|A_0| = 2^{n-1}$ with $n \geqslant 2$. Let N be the normal closure of A_0 in A_{-1}. Since A is of dihedral type, we have $[A_{-1} : N] = [N : A_0] = 2$.

Let G be the free product with amalgamation constructed from the amalgam A. We shall consider the groups A_i, $i = 0, \pm 1$ as subgroups of G. By G_0 we denote the smallest normal subgroup of G containing N and A_1. Then clearly $[G : G_0] = 2$.

Let Γ be the graph whose vertex-set is G/A_{-1} and the edge set is G/A_1; the end-points of an edge zA_1 are the two cosets zA_{-1} and zyA_{-1} ($y \in A_1 - A_0$) which meet zA_1. It is clear that G acts on Γ by left translations and that G is transitive on both vertices and edges of Γ. It can be easily deduced from [7, Théorème 7, p. 49] that Γ is the regular tree of valence four (necessarily infinite). The subgroups A_{-1}, A_1, and A_0 are the stabilizers in G of the vertex A_{-1}, the edge A_1, and the flag (A_{-1}, A_1), respectively. Since A is of dihedral type, the stabilizer A_{-1} of the vertex A_{-1} acts on the four edges incident with this vertex as the dihedral group. In particular, A_{-1} permutes these edges transitively.

Let us say that an edge zA_1, is *even* (resp. *odd*) if $z \in G_0$ (resp. $z \in G - G_0$). An element $z \in G$ preserves or interchanges these two classes of edges according to whether $z \in G_0$ or $z \in G - G_0$. Let us define an *s-arc* as a sequence $(v_0, e_1, v_1, \ldots, e_s, v_s)$ of vertices v_i and edges e_i such that e_i has v_{i-1} and v_i as its end-points and each pair $\{e_i, e_{i+1}\}$ consists of an even and an odd edge.

Then the same argument as in [9, Proof of 7.72, p. 63] can be applied to show that G acts sharply transitively on n-arcs in Γ. I am indebted to G. L. Miller for this observation. In particular, it follows that for $1 \leqslant s \leqslant n$ the fixer in G of an s-arc has order 2^{n-s}.

If n is even (resp. odd) we write $n = 2k$ (resp. $n = 2k + 1$) with $k \geqslant 1$. We fix an $(n+1)$-arc

$$(v_0, e_1, v_1, \ldots, e_{n+1}, v_{n+1})$$

such that $v_k = A_{-1}$, $e_{k+1} = A_1$ if n is odd and $v_{k-1} = A_{-1}$, $e_k = A_1$ if n is even. There is a unique element $z \in G$ such that $zv_i = v_{i+1}$, $0 \leqslant i \leqslant n$, and clearly $z \notin G_0$. Now we define v_i and e_i for all integers i by $v_i = z^i v_0$ and $e_i = z^{i-1} e_1$.

We have $N = A_{-1} \cap G_0$, i.e., N is the subgroup of A_{-1} which preserves the even (and odd) edges. If n is odd we denote by $\hat{v}_i$ the unique nontrivial element of G which fixes all vertices v_j for $|j - i| \leqslant k$. Similarly, if n is even we denote by $\hat{e}_i$ the unique nontrivial element of G which fixes all vertices v_j for $i - k \leqslant j \leqslant i + k - 1$. Then we have

$$N = \langle \hat{v}_0, \hat{v}_1, \ldots, \hat{v}_{n-1} \rangle \quad \text{if } n \text{ is odd,}$$
$$N = \langle \hat{e}_0, \hat{e}_1, \ldots, \hat{e}_{n-1} \rangle \quad \text{if } n \text{ is even.}$$

Since $zv_i = v_{i+1}$ and $ze_i = e_{i+1}$ for all i, we have

$$z\hat{v}_i z^{-1} = \hat{v}_{i+1} \quad \text{resp. } z\hat{e}_i z^{-1} = \hat{e}_{i+1}. \tag{8}$$

Let us put $a_i = \hat{v}_i$ if n is odd and $a_i = \hat{e}_i$ if n is even. Then we have

$$N = \langle a_0, a_1, \ldots, a_{n-1} \rangle, \qquad A_0 = \langle a_1, a_2, \ldots, a_{n-1} \rangle.$$

Furthermore, $B = \langle a_0, a_1, \ldots, a_{n-2} \rangle = z^{-1} A_0 z$ and conjugation by z induces an isomorphism $\sigma: B \to A_0$ such that $\sigma(a_i) = a_{i+1}$, $0 \leqslant i \leqslant n-2$.

If N is abelian let $r = n$. Otherwise let r be the smallest positive integer such that $[a_0, a_r] \neq 1$. Then by a result of Glauberman [5, Lemma 3.5(c)] N is nilpotent of class at most two,

$$Z(N) = \langle a_{n-r}, a_{n-r+1}, \ldots, a_{r-1} \rangle, \tag{9}$$

and $3r \geqslant 2n$, i.e., $r \geqslant m$. Now it is clear that $a_0, a_1, \ldots, a_{n-1}$ satisfy the relations (1), (2), and (3).

Since G is sharply transitive on n-arcs of Γ, there exist unique elements $x \in A_{-1}$ and $y \in A_1$ such that

$$xv_i = v_{n-i-1}, \qquad yv_i = v_{n-i}, \qquad 0 \leqslant i \leqslant n, \tag{10$'$}$$

for n odd, and

$$xv_i = v_{n-2-i}, \qquad yv_i = v_{n-1-i}, \qquad -1 \leqslant i \leqslant n-1, \tag{10$''$}$$

for n even. By (9) we have $a_k \in Z(N)$.

If n is odd then $a_k = \hat{v}_k$ and $v_k = A_{-1}$. Since N is transitive on $2k$-arcs

$$(v_0', e_1', \ldots, e_{2k}', v_{2k}')$$

such that e_k' is even and $v_k' = v_k = A_{-1}$, we conclude that $\hat{v}_k$ fixes all vertices whose distance from v_k is at most k. Since G is transitive on vertices of Γ, it follows that for every vertex v there exists a unique involution $\hat{v} \in G$ which fixes all vertices whose distance from v is at most k. Using this geometric description of involutions $\hat{v}$, we deduce from (10′) that

$$x\hat{v}_i x^{-1} = \hat{v}_{n-i-1}, \qquad y\hat{v}_i y^{-1} = \hat{v}_{n-i}, \qquad 0 \leqslant i \leqslant n.$$

In particular, the relatons (5) and (6) are valid. The symmetry condition (4) follows from (3) and (5).

From (10′) we obtain

$$x^2 v_i = v_i, \qquad 0 \leqslant i \leqslant n-1;$$

$$y^2 v_i = v_i, \qquad 0 \leqslant i \leqslant n.$$

By using sharp transitivity of G on n-arcs, we conclude from here that the relations (7′) are valid.

This completes the proof in the case when n is odd. The case when n is even can be treated similarly; we omit the details.

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