ON FINITE GROUPS CONTAINING THREE CC-SUBGROUPS

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Abstract. A finite group $G$ has a self-centralization system of type $(2|A_1|, 4|A_2|, 4|A_3|)$ if $G$ contains three nonconjugate CC-subgroups $A_1, A_2, A_3$, such that $|N_G(A_1)| = 2|A_1|$, $|N_G(A_2)| = 4|A_2|$, $|N_G(A_3)| = 4|A_3|$. The authors prove that if a finite group $G$ has a self-centralization system of type $(2|A_1|, 4|A_2|, 4|A_3|)$ and $|G| < 3|A_1|^2|A_2|^2|A_3|^2$, then $G$ has a nilpotent normal subgroup $N$ such that $G/N$ is isomorphic to $Sz(q)$ for suitable $q$.

Introduction. A proper subgroup $A$ of a group $G$ is called a CC-subgroup if $C_G(a) < A$ for all $a \in A$. In this paper we shall consider a group $G$ with a set $A_1, A_2, A_3$ of nonconjugate CC-subgroups. If $[N_G(A_i): A_i] = d_i > 1$, $1 < i < 3$, we shall describe this as a self-centralization system of type $(d_1|A_1|, d_2|A_2|, d_3|A_3|)$ or $(d_1, d_2, d_3)$ for short.

The fact is that many simple groups have self-centralization systems. For example, $J_4$, $J_{11}$ and $M_{22}$ have types $(22(23), 28(29), 10(31), 12(37), 14(43))$, $(6(7), 10(11), 6(19))$ and $(4(5), 3(7), 5(11))$, respectively. Groups of Ree type have type $(6, 6)$.

Several authors have studied groups with self-centralization systems of type $(2)$. (See [6], [7] and [8].) It is an open question of Higman [7] whether or not there are finite simple groups of type $(2, 4, 4)$ other than the Suzuki groups. Usami [10] answers this question under restrictions on the order of the CC-subgroups of $G$ and the assumption that if $r$ is an odd prime then an $S_r$-subgroup of $G$ is abelian and $|G| < 3|A_1|^2|A_2|^2|A_3|^2$.

In this paper we shall prove the following theorem:

**Theorem A.** If $G$ is a finite simple group with a self-centralization system of type $(2|A_1|, 4|A_2|, 4|A_3|)$ and $|G| < 3|A_1|^2|A_2|^2|A_3|^2$, then $G$ is isomorphic to one of the Suzuki groups.

Theorem A has the following corollary.

**Corollary A.** If $G$ is a finite group with a self-centralization system of type $(2|A_1|, 4|A_2|, 4|A_3|)$ and $|G| < 3|A_1|^2|A_2|^2|A_3|^2$, then $G$ has a nilpotent normal subgroup $N$ such that $G/N$ is isomorphic to $Sz(q)$ for suitable $q$.

Many papers appear in the literature which classify groups which have a character $\Lambda$ of small degree relative to the orders of some subgroups of $G$. In a forthcoming paper, the authors will use Theorem A to classify all groups of type...
(2n_1, 4n_2, 4n_3) which have a character Λ of relatively small degree compared to \( n_1n_2n_3 \).

1. Our notations are taken from [5] and character notation from [6]. If \( K \) is a nonempty set, let \(|K|\) denote the number of elements in \( K \). If \( p \) is a prime, then \( G_p \) denotes a Sylow \( p \) subgroup of \( G \).

Let \( G \in Hypothesis\ A \) if \( G \) is a finite simple group with a self-centralization system of type \((2|A_1|, 4|A_2|, 4|A_3|)\).

Let \(|A_1| = n_1, |A_2| = n_2 \) and \(|A_3| = n_3 \) where \( n_2 < n_3 \). Let \( t_1 = (n_1 - 1)/2, t_2 = (n_2 - 1)/4 \) and \( t_3 = (n_3 - 1)/4 \).

Let \( G \in Hypothesis\ A \), then Theorem 5.1 [7] implies \( G \) has one class of involutions. If \( τ \) is a fixed involution, then by conjugacy we may choose \( A_1, A_2, A_3 \) to be normalized by \( τ \).

**Lemma 1.** Let \( G \in Hypothesis\ A \), then one of the following holds:

(i) \( G \) is isomorphic to \( Sz(q) \).

(ii)(a) \((n_1n_2n_3, 3) = 1 \).

Proof. Assume \( G \) is not isomorphic to a Suzuki group; then [4] and [3] imply \( G \) contains an element of order 6. Hence \( A_i \) a CC-subgroup for \( i = 1, 2, 3 \) implies \((n_1n_2n_3, 3) = 1 \). Walter [11] implies \( G \) contains an element of order 4.

Let \( G \in Hypothesis\ B \) if \( G \in Hypothesis\ A \), \( G \) is not isomorphic to a Suzuki group, and \(|G| < 3|A_1|^2|A_2|^2|A_3|^2 \).

Assume \( G \in Hypothesis\ A \), then \( n_1 > 3 \). Hence, there are \( t_i \) irreducible characters \( X_i, i = 1, \ldots, t_i \), of \( G \) which are associated with \( A_i \) and one nonprincipal irreducible character \( Y \) such that \( Y(z) \neq 0 \) for \( z \in A_i \) [6]. Let \( 1_G \) denote the principal character of \( G \). Lemma 3 [6] implies \( X_i(1) = kn_1 + 2ε \) and \( Y(1) = kn_1 + ε \) where \( k \) is a positive integer and \( ε = ±1 \). Lemma 6 [6] implies \(|G| = n_1X_1(1)Y(1)(ln_1 + 1) \) where \( l \) is a nonnegative integer.

We incorporate some properties of \( X_1 \) and \( Y \) into Lemma 2.

**Lemma 2.** Assume \( G \in Hypothesis\ A \) and let \( T = \bigcup_{x \in G} (A_x^T)^2 \), then the following hold:

(i) If \( x \in G - T \), then \( X_1(x) \) and \( Y(x) \) are integers, and \( X_1(x)Y(x) \) is a nonnegative integer.

(ii) \( \sum_{x \in G - T} X_1(x)Y(x) = |G|/n_1 \).

(iii) \( |CG(τ)|^2 = (ln_1 + 1)(Y(1) - Y(τ))^2 \).

(iv) \( |Y(τ)| < k + ε \).

(v) \( Y(1) - Y(τ) ) > k^2(n_1 - 1)^2 \).

(vi) \( n_2n_3, X_1(1) \neq 1. If l = 0, then n_2n_3, X_1(1) \).

Proof. Since \( Y \) is the only nonprincipal nonexceptional irreducible character of \( G \) which does not vanish on \( A_x^T \), \( Y \) is equal to all its algebraic conjugates. Hence \( Y(z) \) is an integer for all \( z \in G \). Lemma 3 [6] implies if \( x \in G - T \), then \( X_1(x) = Y(x) + ε \). Hence \( X_1(x) \) is an integer. Now \( X_1(x)Y(x) = X_1(x)Y(x) + εY(x) \) and \( Y(x) \) an integer implies (i).
The orthogonality relations imply $\sum G X_1(g) Y(g) = 0$. Now $\sum_T X_1(g) Y(g) = (|G|/2n_1)\sum G \xi_1(g)$ where $\xi_1$ is an irreducible character of $N_G(A_i)$ associated with $X_1$. $\sum_T \xi_1(g) = -2$; hence $\sum G X_1(x) Y(x) = |G|/n_1$ and (ii) is proved.


$N_G(A_i)$ a Frobenius group of order $2n_1$ implies there are $t_1$ irreducible characters $\xi_i$ of $N_G(A)$ such that $\xi_1(1) = 2$ and $\xi_i$ vanishes off $A_i$. Let $\lambda$ be a faithful linear character of $N_G(A_i)/A_i$, then $\lambda, \lambda^2, \xi_i[i = 1, \ldots, t_1]$ are all the irreducible characters of $N_G(A_i)$. Since $Y$ is a nonexceptional character, it follows that $Y|_{N_G(A_i)} = r\lambda^2 + s\lambda + \sum_{j=1}^{t_1} k_j \xi_j$ where $r + s = k + e$. Now $\tau \in N(A_i)$ implies $Y(\tau) = r - s$; hence $|Y(\tau)| \leq k + e$. Parts (iv) and (v) of the lemma follow.

Now $\sum A_i X_1(g)X_1(g) = e^2\sum A_i \xi_1(g)\xi_1(g) = 2n_1 - 4$. Thus $(X_1, X_1) = |G|$ implies $\sum G X_1(x)X_1(x) + |G|(2n_1 - 4)/n_1 = 2|G|/n_1$. Suppose $(X_1(1), n_2n_3) = 1$. $X_1$ is a nonexceptional character for $i = 2$ or 3. Hence, if $t_i > 2$ for $i = 2$ or 3, it follows that $|X_1(x)| = 1$ for all $x \in A_i$. If $t_2 = 1$, then $n_2 = 5$ and all elements of order 5 are again conjugate. Hence (i) implies $|X_1(x)| = 1$ for all $x \in A_i$. Thus $(n_2n_3, X_1(1)) = 1$ implies $|G|(n_2 - 1)/4n_2 + |G|(n_3 - 1)/4n_3 < |G|/2n_1$.

Now $n_1 > 5$ implies $1/10 < 1/2 - 2/n_1 < 1/4n_2 + 1/4n_3 < 2/4n_2$. Hence $n_2 < 5$, which is a contradiction. Therefore, $(X_1(1), n_2n_3) \neq 1$.

If $l = 0$, then $|G_2| |X_1(1)|$ or $|G_2| |Y(1)|$. Theorem 17.4 [4] implies $X_1(\tau) = 0$ or $Y(\tau) = 0$. Lemma 3 [6] implies $Y(1) - Y(\tau) = X_1(1) - X_1(\tau)$, hence (iii) implies $|C_G(\tau)| = X_1(1)$ or $Y(1)$. Now $(n_2n_3, X_1(1)) \neq 1$ implies $|C_G(\tau)| = Y(1)$ so that $n_2n_3|X_1(1)$.

**Lemma 3.** Assume $G \in$ Hypothesis B, then $n_1 > 11$ and $n_2n_3|X_1(1)$.

**Proof.** Suppose $n_1 > 7$, then Lemma 1(ii)(a) implies $n_1 > 11$. Suppose $t_i > 2$ for $i = 2$ or 3. Then $\{X_j|j = 1, \ldots, t_1\}$ does not contain any exceptional characters associated with $N(A_i)$. It follows that $X_i(z) = X_i(1)$ for $z \in A_i$. Since $X_i(1) = X_i(z)$ for $j = 1, \ldots, t_1$, the orthogonality relations imply $5X_1(1) < t_1X_i(1) < |C_G(z)| = 1 - 4$. Lemma 2(i) implies $X_1(1) = 0$. Thus $n_1 > 7$ implies $n_2n_3|X_1(1)$.

We will assume $n_1 < 7$ and obtain a contradiction. Let $x_1, i = 1, \ldots, t_1$, be representatives in $A_i$ of the conjugacy classes of $A_i$. Similarly let $y_j, j = 1, \ldots, t_2$, and $z_k, k = 1, \ldots, t_3$, be representatives of the conjugacy classes of $A_2$ and $A_3$. Let $B = \{x_1, y_1, z_k|i, j, k\}$, then $|B| = b = t_1 + t_2 + t_3$.

If $C_G(\tau)uC_G(\tau) \cap C_G(\tau)vC_G(\tau) \neq \emptyset$ for distinct elements $u, v \in B$, then there are elements $w$ and $h \in C_G(\tau)$ such that $v = huw$. Conjugation by $\tau$ yields $v^{-1} = v^* = (huv)^* = h^{-1}u$. It follows that $v = (v^*)^{-1} = (h^{-1}u)^{-1} = w^{-1}uh^{-1}$. Hence, $v^2 = (w^{-1}uh^{-1})(huv) = w^{-1}u$. Since all elements in $B$ have odd order, $v^2 = u^2$ implies $|\langle u \rangle| = |\langle v \rangle| = k$. Now $k$ odd implies $v = (u^2)(k + 1)/2 = (u^2)(k + 1)/2 = u^2(k + 1)/2 = u^2$. This contradicts the fact that $u$ and $v$ are not conjugate. Therefore, $C_G(\tau)uC_G(\tau) \cap C_G(\tau)vC_G(\tau) = \emptyset$ if $u$ and $v$ are distinct. If $w \in C_G(\tau) \cap C_G(\tau)^v$ where $u \in B$, then $w \in C_G(\tau) \cap C_G(\tau)^v = C_G(\tau) \cap C_G(u^2)$.
Therefore \(|C_G(\tau)uC_G(\tau)| = |C_G(\tau)|^2\) if \(u \in B\). It follows that \(b|C_G(\tau)|^2 < |G|\).

Lemma 6 \([6]\) implies that \(|G|(Y(1) - Y(\tau))^2 = n_1(kn_1 + \varepsilon)(kn_1 + 2\varepsilon)|C_G(\tau)|^2\).

Hence, \(b|C_G(\tau)|^2 < |G|\) implies that \(b(Y(1) - Y(\tau))^2 < n_1(kn_1 + \varepsilon)(kn_1 + 2\varepsilon)\).

Lemma 2(v) now implies

\[bk^2(n_1 - 1)^2 < n_1(kn_1 + 2\varepsilon)(kn_1 + \varepsilon).\] (3.1)

Lemma 2(iii) implies that \(n_2n_3|X_1(1)Y(1)|\).

Suppose \(n_1 = 5\). Lemma 1 implies \(n_2 > 13\) and \(n_3 > 17\), hence \(b > 9\). Inequality (3.1) implies \(19k^2 - 75ke - 10 < 0\). Hence \(\varepsilon = 1\) and \(k < 4\). Direct computation of \(X_1(1)Y(1)\) for \(\varepsilon = 1\) and \(1 < k < 4\) imply there are no \(n_2\) and \(n_3\) such that \(n_2n_3|X_1(1)Y(1)|\). Hence \(n_1 > 5\).

Suppose \(n_1 = 7\). If \(k < 3\), then direct computation for \(\varepsilon = \pm 1\) implies there are no \(n_2\) and \(n_3\) such that \(n_2n_3|X_1(1)Y(1)|\). Hence we may assume \(k > 4\). Inequality (3.1) implies \(b < 11\), hence \((n_2 - 1)/4 + (n_3 - 1)/4 < 11 - 3 = 8\). Calculation implies \((n_2, n_3) = (5, 13), (5, 17)\) or \((13, 17)\).

If \((n_2, n_3) = (13, 17)\), then \(b = 10\) and (3.1) implies \(17k^2 - 147ke - 14 < 0\).

Hence \(\varepsilon = 1\) and \(k < 9\). Direct computation implies \(13 \cdot 17|X_1(1)Y(1)|\) for \(\varepsilon = 1\) and \(k < 9\). Hence \((n_2, n_3) = (5, 13)\) or \((5, 17)\).

Suppose \(l \neq 0\). Lemma 2(vi) implies \(l^2 + 1\) is a square, hence \(l > 5\). Let \(T \in \{X_1, Y\}\) where \(n_3|T(1)\); then \(T(1) = fn_3\). Since \(G\) satisfies Hypothesis B, \(|G| < 3 \cdot 7^2 \cdot 5^2 \cdot n_3^2\). Now \(3 \cdot 5^2 \cdot 7^2 \cdot n_3^2 > |G| > 7(fn_3 - 1)^2(5 \cdot 7 + 1)\) implies \(f < 4\). If \(n_3 = 17\) and \(T = Y\), then \(17f = \pm 1 \mod 7\) implies \(f = 2\) and \(Y(1) = 34 = 5 \cdot 7 - 1\). Hence \(X_1(1) = 33\) and \(5|Y(1)X_1(1)|\). If \(n_3 = 17\) and \(T = X_1\), then \(17f = \pm 2 \mod 7\) implies \(f = 3\) and \(X_1(1) = 7 \cdot 7 + 2\). Hence \(Y(1) = 50\), but \(n_2 = 5\) implies \(50|G|\). Therefore \(n_3 \neq 17\). Similar calculations imply \(n_3 \neq 13\).

Hence, we may assume \(l = 0\). Lemma 2(iii) implies \(X_1(1) = en_2n_3\). Now \(7(en_2n_3 - 1)(en_2n_3) < |G| < 3 \cdot 7^2 \cdot 5^2 \cdot n_3^2\) implies \(e < 4\). Since \(4|G|\) and \(|G|\) \(|Y(1)|\), \(e = 1\) or \(3\). However \(17 \cdot 5 = 1 \mod 7\) and \(3 \cdot 17 \cdot 5 = 3 \mod 7\) imply \(n_3 \neq 17\). If \(n_3 = 13\), then \(5 \cdot 13 \cdot 3 = 6 \mod 7\) implies \(X_1(1) = 5 \cdot 13\) and \(Y(1) = 64\). Hence \((|G|, 3) = 1\) and \(G \notin\) Hypothesis B.

**Lemma 4.** Assume \(G \in\) Hypothesis B. Let \(U = \{Z|Z\text{ irreducible character of }G\text{ and }n_1n_2n_3Z(1)\}\), then \(|U| < 4 + t_1 + t_2 + t_3\).

**Proof.** If \(n_2 > 5\), Lemma 4 [10] and Lemmas 1 and 3 imply the result. Hence we may assume \(n_2 = 5\). Let \(x_i, i = 2, 3\), denote elements of \(A_5^f\) and \(A_5^g\) respectively. Lemma 3 implies \(n_2n_3|X_1(1)|\). Lemma 3 [6] implies \(Y(x_i) = -\varepsilon\) for \(i = 2, 3\). Since all elements of order 5 are conjugate, the orthogonality relations imply there are 3 irreducible characters \(Z_1, Z_2\) and \(Z_3\) of \(G\) such that

\[|Z_i(x_2)| = 1.\] (4.1)

If \(V = \{1_G, Y, Z_1, Z_2, Z_3\}\), then \(|V| = 5\) and \(V\) contains all the irreducible characters of \(G\) whose degrees are not divisible by 5.

Let \(\{x_3\} = 1, \ldots, t_3\) be the exceptional characters of \(G\) associated with \(N_G(A_3)\). The orthogonality relations imply there are 2 irreducible characters \(Z_5, Z_4\)
of $G$ such that $|Z_5(x_3)| = |Z_4(x_3)| = 1$. If $W = \{1_G, Y, Z_4, Z_5, X_{3j} | j = 1, \ldots, t_3\}$, then $|W| = t_3 + 4$ and $W$ contains all the irreducible characters of $G$ whose degrees are not divisible by $n_3$.

Now $Y(x_2)Y(x_3) + 1_G(x_2)1_G(x_3) = 2$ and the orthogonality relations imply

$$\sum_{i=1}^{3} Z_i(x_2)Z_i(x_3) = -2.$$  \hspace{1cm} (4.2)

Hence $Z_i(x_3) \neq 0$ for some $i = 1, 2, 3$. Now $X_{3j}(x_2) = X_{3,1}(x_2)$ for $j = 1, \ldots, t_3$; hence $\{Z_j | i = 1, 2, 3\} \cap \{X_{3j} | j = 1, \ldots, t_3\} \neq \emptyset$ implies $(n_3 - 1)/4 = 3$ and $\{Z_j | i = 1, 2, 3\} = \{X_{3j} | j = 1, 2, 3\}$. Hence $U = \{1_G, Y, X_i, X_{3j}, Z_4, Z_5 | i = 1, \ldots, t_1, j = 1, \ldots, t_3\}$ and $|U| = t_1 + t_2 + t_3$. If $\{Z_j | i = 1, 2, 3\} \cap \{X_{3j} | j = 1, \ldots, t_3\} = \emptyset$, then (4.1) and (4.2) imply $\{Z_4, Z_5\} \subset \{Z_1, Z_2, Z_3\}$. Hence $U = \{1_G, Y, Z_1, Z_2, Z_3, X_i, X_{3j} | i = 1, \ldots, t_1, j = 1, \ldots, t_3\}$ so that $|U| = t_1 + t_2 + t_3 + 4$.

**Lemma 5.** Assume $G \in$ Hypothesis B, then $|G| > (n_1n_2n_3)^2$. If $r$ is an odd prime which is relatively prime to $3n_1n_2n_3$ and $r | |G|$, then $r = 5$ and $G_5$ is an elementary abelian CC-subgroup of order 25.

**Proof.** Let $t$ be the number of conjugacy classes of elements in $G$ which do not meet $(A_1 \cup A_2 \cup A_3)^\circ$. Lemma 1(ii) implies $t = 5 + j$ where $j > 0$. Lemma 4 implies there are at least $j + 1$ irreducible characters $Z$ of $G$ such that $n_1n_2n_3 | Z(1)$. Hence $(j + 1)(n_1n_2n_3)^2 < |G|$ implies $|G| > (n_1n_2n_3)^2$ and $j < 1$.

Let $r$ be as in the hypothesis, then $j = 1$. Hence, centralizers of elements of order $r$ are $G_r$ groups. Therefore, $G_r$ is a CC-subgroup of $G$ [9]. Since all elements of $G_r^\circ$ are conjugate, $G_r$ is elementary abelian, and $C_{G_r}(G_r) = G_r$. Now $j = 1$ and Lemma 1 imply $G$ contains no elements of order 12 or 8.

$N_G(G_r)$ is a Frobenius group with $G_r$ as a Frobenius kernel. Let $L$ be the Frobenius complement. Clearly $|L| = |G_r^\circ|$. Since $L$ is a Frobenius complement, an $S_3$-subgroup of $L$ is cyclic or generalized quaternion and the Sylow subgroups of $L$ of odd prime order are cyclic. In this case, the highest power of 2 dividing $|L|$ is 2 or 4 or 8 by Lemma 1. If an odd prime $r'$ divides $|L|$, it is well known that $2 | |Z(L)|$. Hence $r' = 3$. Hence $|L| = 2, 4, 8, 6, 12, \text{ or } 24$. Consequently, $|G_r| = 3, 5, 9, 7, 13, \text{ or } 5^2$. The hypothesis implies $(|G_r|, 3) = 1$; hence $|G_r| = 5, 7, 13, 5^2$.

On the other hand, let $y$ be an element of order $r$. Then $N_G(\langle y \rangle)/C_G(y)$ is a cyclic group of order $r - 1$ since the elements of order $r$ form a single class. Therefore, the possibilities of the order of a cyclic subgroup of even order in $G$ are $2, 4, 6$ and $|G_r| = 5, 7, \text{ or } 5^2$. Let $x$ be an element of order $r$. The class equation implies

$$|G| > 1 + \frac{|G|}{|G_r|} + \frac{|G|(n_1 - 1)}{2n_1} + \frac{|G|(n_2 - 1)}{4n_2} + \frac{|G|(n_3 - 1)}{4n_3}.$$  \hspace{1cm} (4.1)

Hence $1/2n_1 + 1/4n_2 + 1/4n_3 > 1/|G_r|$. Lemma 3 implies $n_1 > 11$. Now $n_2 > 5$ and $n_3 > 13$ imply $|G_r| > 7$. 

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Lemma 6. Assume $G \in \text{Hypothesis B}$, then the following hold:

(i) $X_1(1) = n_2n_3$.
(ii) $ln_1 + 1 = n_1 + 1$ or $3n_1 + 1$.
(iii) $ln_1 + 1 = 2^{2a}3^{2b}$ where $a$ is a positive integer and $b$ is a nonnegative integer.

Proof. Lemma 3 implies $X_1(1) = en_2n_3$, where $e$ is a positive integer.

Suppose $l = 0$. As in the proof of Lemma 2, we see that $|C_G(\tau)| = Y(1)$. Lemma 5 implies $(n_1n_2n_3)^3 < n_1(en_2n_3)(en_2n_3 + 1)$, hence $e > 1$. Lemma 1 implies $3| |C_G(\tau)|$, hence $(X_1(1), Y(1)) = 1$ implies $|G_3| |Y(1)|$. Therefore $X_1(1) = en_2n_3$ where $e$ is odd, $(e, 3n_1n_2n_3) = 1$, and $e > 1$. Lemma 5 implies $(|C_G(\tau)|, 5) = 5$ and $X_1(1) = 25n_2n_3$. Hence $X_1(1) = 1$ (mod 4). Since $4|Y(1)$, $Y(1) = X_1(1) - e$ implies $e = 1$. This contradicts Theorem A [2]. Hence $l > 1$.

Since $l \neq 0$,

$$n_1(en_2n_3)(en_2n_3 - 1)(ln_1 + 1) < |G| < 3(n_1n_2n_3)^2$$

implies $le^2 < 4$. Hence $e = 1$ and $l < 3$. Lemma 2(iii) implies $ln_1 + 1$ is a square, hence $l = 1$ or 3. Lemmas 1, 5 and 2(iii) imply $ln_1 + 1 = 2^{2a}3^{2b}$, where $a > 1$.

Lemma 7. Assume $G \in \text{Hypothesis B}$, then $ln_1 + 1 = 2^{2a}3^{2b}$ where $a$ and $b$ are positive integers.

Proof. Lemma 6(iii) implies $a$ is a positive integer. Suppose $b = 0$. Lemmas 1 and 5 imply $|C_G(\tau)| = |G_2|3^c$ where $c > 1$. Lemma 6 implies $X_1(1) = n_2n_3$. Hence $ln_1 + 1 = 2^{2a}$ implies $|G_3| |Y(1)|$. Lemma 2(iii) implies $(Y(1))^2(ln_1 + 1)3^{2c} = |C_G(\tau)|^2 = (ln_1 + 1)(Y(1) - Y(\tau))^2$. Valuation Theory implies $|Y(\tau)| = Y(1)2$ and $|Y(\tau)|_3 = 3^c$. Lemma 2(iv) now implies

$$|C_G(\tau)| = |Y(\tau)|_2 Y(\tau)|_3(ln_1 + 1) < |Y(\tau)|(ln_1 + 1) < (k + 1)(ln_1 + 1)$$

Lemma 2(iii) and (v) imply $|C_G(\tau)|^2 > k^2(n_1 - 1)^2(ln_1 + 1)$. It follows that

$$(k + 1)^2(ln_1 + 1)^2 > |C_G(\tau)|^2 > k^2(n_1 - 1)^2(ln_1 + 1)$$

Now $l < 3$ and $k > 1$ imply $4(3n_1 + 1) > (n_1 - 1)^2$. Hence $n_1 < 15$. Lemmas 1 and 3 imply $n_1 = 11$ or 13. However $n_1 + 1 \neq 2^a$ if $l = 1$ or 3 and $n_1 = 11$ or 13.

Proof of Theorem A. We will assume $G \in \text{Hypothesis B}$ and obtain a contradiction. Lemma 7 implies $|G_2| |Y(1)|$, hence Theorem 17.4 [1] implies there is an element $y$ such that $|y\bar{y}| = 2^k > 1$ and $Y(y) \neq 0$. Since $Y$ is integer valued, $Y(y') = Y(y)$ if $(i, 2) = 1$. Lemma 6 implies $X_1(1) = n_2n_3$. Lemma 2 implies $X_1(z)$ is an integer for $z \in G - \bigcup g(A_1)^g$. Hence $(n_2n_3, 2) = 1$ implies $X_1(y') \neq 0$ for any $i$.

Let $R = \{y^g \mid g \in G\}$, $n = |G|/|G_2|$ and $U_R = \{x \mid x^n \in R\}$. Clearly $\bigcup_R$ is a union of conjugacy classes. Since $(n, 8) = 1$, there is an odd integer $i_0$ such that $y^{i_0} = y$. Let $z_1 = y^{i_0}$. Lemma 2(i) implies $X_1(z_1)Y(z_1)$ is a positive integer. Let $z_1, z_2, \ldots, z_t$ be representatives of the conjugacy class in $\bigcup_R$. Now $X_1Y$ is a character of $G$, hence Theorem 17.3 [1] implies

(a) $(|G|, n)^{-1}\sum_{y^g} X_1(x)Y(x) = c$ where $c$ is an algebraic integer. Since $\bigcup_R < G - T$, Lemma 2(i) and $X_1(z_1)Y(z_1) \neq 0$ imply $c$ is a positive rational number. Hence $c$ is a positive integer.
ON FINITE GROUPS CONTAINING THREE CC-SUBGROUPS

\[ |G_2| = Y(1)^2(ln_1 + 1)^2 \] and (a) imply (b) \( \sum \_{x \in G} X_1(x)Y(x) = c(|G|, n) = c|G|/|G_2| = c|G|/Y(1)^2(ln_1 + 1)^2 \). Now

\[ \sum \_{x \in G} X_1(x)Y(x) = \sum \_{i=1}^{t} \frac{X_1(z_i)Y(z_i)|G|}{|C_G(z_i)|} \]

and the orthogonality relations imply

\[ Y(1) \left| \frac{X_1(z_i)Y(z_i)|G|}{|C_G(z_i)|} \right| \quad \text{for } i = 1, \ldots, t. \]

Equation (b) now implies

\[ Y(1) \left| \frac{c|G|}{Y(1)^2(ln_1 + 1)^2} \right| = \frac{c|G|}{|G_2|}. \]

Hence \( c = Y(1)^2d \) where \( d \) is a positive integer. Thus

\[ \sum \_{x \in G} X_1(x)Y(x) = dG/(ln_1 + 1)^2. \]

Lemma 2(ii) now implies \( dG/(ln_1 + 1)^2 < |G|/n \), so that \( (ln_1 + 1)^2 > n \). Lemma 7 implies \( l > 3 \) which contradicts Lemma 6.

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