TAME MEASURES ON CERTAIN COMPACT SETS

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Abstract. A finite complex Borel measure $\mu$ on a compact subset $X \subset \mathbb{C}^n$ is called tame if there exist finite measures $\sigma_1, \ldots, \sigma_n$ on $X$ with

$$\int_X \phi \, d\mu = \int_X \sum_{j=1}^n \frac{\partial \phi}{\partial z_j} \, d\sigma_j$$

for every $\phi \in C_0^\infty(\mathbb{C}^n)$. We define $X_T = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1 \text{ and } z_1 \in T\}$, where $T$ is a compact subset of $\{|z| < 1\}$ in $\mathbb{C}^1$. It is shown in this paper that tame measures form a weak-* dense subset of $L^1(X_T)$. It follows then, with the help of a theorem by Weinstock, that $R(X_T)$ is a local algebra.

Let $X$ be a compact set in $\mathbb{C}^n$. $C(X)$ is the algebra of all continuous functions on $X$. $R_0(X)$ is the algebra of all rational functions $P/Q$ on $\mathbb{C}^n$ with $P, Q$ polynomials and $Q \neq 0$ on $X$. $R(X)$ is the uniform closure of $R_0(X)$ in $C(X)$.

It is a well-known consequence of Cauchy-Green formula that if $\mu$ is a complex Borel measure with compact support $X \subset \mathbb{C}^1$, then

$$\int_X \phi \, d\mu = -\frac{1}{2\pi i} \int \frac{\partial \phi}{\partial \bar{z}} \left( \int \frac{1}{\bar{z} - z} \, d\mu(z) \right) \, dz \wedge d\bar{z} \quad (1)$$

holds for every $\phi \in C_0^\infty(\mathbb{C})$. It follows that $\mu$ is an orthogonal measure for $R(X)$ iff

$$\hat{\mu} = \int \frac{1}{\bar{z} - z} \, d\mu(z) \quad (2)$$

is supported on $X$, or, equivalently, the measure $\hat{\mu}(z)dz \wedge d\bar{z}$ is supported on $X$. This gives a description of orthogonal measures for $R(X)$ where $X \subset \mathbb{C}$. While no general description for measures on $X \subset \mathbb{C}^n$, $n > 1$, orthogonal to $R(X)$ is available, we introduce the following definition.

Definition. Let $X$ be a compact set in $\mathbb{C}^n$. A finite complex Borel measure is tame if there exist finite measures $\sigma_1, \ldots, \sigma_n$ on $X$ with

$$\int_X \phi \, d\mu = \int_X \sum_{j=1}^n \frac{\partial \phi}{\partial z_j} \, d\sigma_j \quad \text{for every } \phi \in C_0^\infty(\mathbb{C}^n). \quad (*)$$

(1) and (2) now imply that for $X \subset \mathbb{C}$ a measure $\mu$ on $X$ is orthogonal to $R(X)$ iff $\mu$ is tame.

Let $X \subset \mathbb{C}^n$ be fixed. Suppose that a tame measure $\mu$ exists on $X$ with $\mu \neq 0$. If $\phi \in R_0(X)$ then $\partial \phi/\partial z_j \equiv 0$ on $X$ for all $j$. So by $(*) \int_X \phi \, d\mu = 0$. It follows that $\mu \perp R(X)$ and hence $R(X) \neq C(X)$. Thus the existence of tame measures imply that $R(X) \neq C(X)$.

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In this paper, we restrict ourselves to subsets of $\partial B = \{(z_1, z_2): |z_1|^2 + |z_2|^2 = 1\}$ which have the form $X_T = \{(z_1, z_2) \in \partial B: z_1 \in T\}$ where $T$ is a compact subset of \{|z_1| < 1\} in $\mathbb{C}$. We study orthogonal measures for $R(X_T)$ and problems related to these measures.

Basener [1] has constructed a compact subset $\tilde{X}_T$ (which has the above stated form) of $\partial B$ such that $\tilde{X}_T$ is rationally convex, yet $R(\tilde{X}_T) \neq C(\tilde{X}_T)$. In the following, we will construct an ample family of tame measures for $R(X_T)$, provided that $R(T) \neq C(T)$. In fact, they form a weak-* dense set of $R(X_T)$. This gives an alternative explanation why $R(\tilde{X}_T) \neq C(\tilde{X}_T)$. Moreover, the weak-* density along with a theorem of Weinstock [2] lead to the conclusion that $R(X_T)$ is a local algebra. I.e., if $\{U_\alpha\}$ is a finite open covering of $X_T$ and if $f \in C(X_T)$ is such that $f|_{X_T \cap U_\alpha}$ is in $R(X_T \cap U_\alpha)$, for all $\alpha$, then $f \in R(X_T)$. The main results may be stated as follows.

**Theorem 1.** Assume that $R(T) \neq C(T)$. Then the set of tame measures on $X_T$ is weak-* dense in the set of all orthogonal measures to $R(X_T)$ on $X_T$.

**Theorem 2.** Let $\phi$ be a smooth function with $\partial \phi / \partial z_i \equiv 0$ on $X_T$, $i = 1, 2$. Then $\phi \in R(X_T)$.

**Theorem 3.** $R(X_T)$ is a local algebra.

**Notations.**

\[ B = \{(z_1, z_2): |z_1|^2 + |z_2|^2 < 1\}, \]
\[ \partial B = \{(z_1, z_2): |z_1|^2 + |z_2|^2 = 1\}, \]
\[ \Delta = \{z_1 \in \mathbb{C}: |z_1| < 1\}, \]
\[ X_T = \{(z_1, z_2) \in \partial B, z_1 \in T\} \text{ where } T \text{ is a compact subset of } \Delta, \]
\[ \Gamma_{z_1} = \{(z_1, (1 - z_1 \bar{z}_1)^{1/2} e^{i\theta}): -\pi < \theta < \pi\}. \]

Let $\phi$ be any smooth function in a neighborhood of $X_T$. Let $\tilde{\phi}$ denote the composite $\phi \circ p$ where $p$ is the map from \{|z_1| < 1\} $\times$ $[-\pi, \pi]$ to $\partial B$ defined by $p(z_1, \theta) = (z_1, (1 - z_1 \bar{z}_1)^{1/2} e^{i\theta})$. For each fixed $z_1 \in T$, $\tilde{\phi}$ has the following Fourier expansion on $\Gamma_{z_1}$:

\[ \phi(z_1, z_2) = \tilde{\phi}(z_1, \theta) = \sum_{n=-\infty}^{\infty} \phi_n(z_1) e^{int} e^{i\theta}, \quad z_2 = (1 - z_1 \bar{z}_1)^{1/2} e^{i\theta}, \]

where

\[ \phi_n(z_1) = \int_{-\pi}^{\pi} \phi(z_1, t) e^{-int} \frac{dt}{2\pi} \]

is the $n$th Fourier coefficient of $\tilde{\phi}(z_1, \theta)$.

It is well known that

(i) $\phi_n(z_1)$ is smooth in $z_1$,

(ii) if $n \neq 0$, $|\phi_n(z_1)| \leq M/n^3$ for all $z_1 \in T$ where $M$ is a constant depending on $\phi$. 

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THEOREM 1. Assume that \( R(T) \neq C(T) \). Then the set of tame measures on \( X \) is weak-* dense in the set \( R(X_T) \perp \) of all measures on \( X_T \) orthogonal to \( R(X_T) \).

PROOF. Let \( \nu \) be a nonzero orthogonal measure for \( R(T) \). Consider the linear functional which assigns to each \( f \) in \( C(X_T) \) the value

\[
\int_T \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} f(z_1, z_2) \frac{dz_2}{z_2} \right) d\nu(z_1).
\]

Since \( T \) is a compact subset of \( \Delta \), we have \( X_T \cap \{ z_2 = 0 \} = \phi \). Hence the above is well defined. It is easy to see that this linear functional is continuous, therefore it defines a measure \( \mu \) on \( X_T \), i.e.

\[
\int f \, d\mu = \int \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} f(z_1, z_2) \frac{dz_2}{z_2} \right) d\nu(z_1) \quad \text{for all } f \in C(X_T).
\]

Assertion. \( \mu \) is tame.

Let \( \phi \in C_0^\infty(C^2) \),

\[
\int_{X_T} \phi \, d\mu = \int \left( \frac{1}{2\pi i} \int_{\Gamma_{z_1}} \phi(z_1, z_2) \frac{dz_2}{z_2} \right) d\nu(z_1)
= \int_T \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\phi}(z_1, \theta) \, d\theta \right) d\nu(z_1)
= \int_T \phi_0(z_1) \, d\nu(z_1)
= \frac{-1}{2\pi i} \int \frac{\partial \phi_0}{\partial \xi_1} \tilde{\nu}(\xi_1) \, d\xi_1 \wedge d\bar{\xi}_1 \quad \text{by (1)}.
\]

Assume the following lemma which will be proved later.

LEMMA.

\[
\frac{\partial \phi_0}{\partial \xi_1} = \left( \frac{\partial \phi}{\partial \xi_1} - \frac{\xi_1}{\xi_2} \frac{\partial \phi}{\partial \xi_2} \right)_0,
\]

the zeroth Fourier coefficient of \( \frac{\partial \phi}{\partial \xi_1} - (\xi_1/\xi_2)(\partial \phi/\partial \xi_2) \).

We get that

\[
\int_{X_T} \phi \, d\mu = \frac{-1}{2\pi i} \int \left( \frac{\partial \phi}{\partial \xi_1} \frac{\xi_1}{\xi_2} \frac{\partial \phi}{\partial \xi_2} \right) \tilde{\nu}(\xi_1) \, d\xi_1 \wedge d\bar{\xi}_1
= \frac{-1}{2\pi i} \int \left[ \frac{1}{2\pi i} \int_{\Gamma_{z_1}} \left( \frac{\partial \phi}{\partial \xi_1} \frac{\xi_1}{\xi_2} \frac{\partial \phi}{\partial \xi_2} \right) \frac{dz_2}{z_2} \right] \tilde{\nu}(\xi_1) \, d\xi_1 \wedge d\bar{\xi}_1.
\]

Let \( \sigma_1 \) be the measure on \( \partial B \) such that for \( f \) in \( C(\partial B) \),

\[
\int f(z_1, z_2) \, d\sigma_1 \equiv \int \frac{1}{4\pi^2} \left( \int_{\Gamma_{z_1}} f(z_1, z_2) \frac{dz_2}{z_2} \right) \tilde{\nu}(z_1) \, dz_1 \wedge d\bar{z}_1
\]

and let \( \sigma_2 = -(z_1/z_2)\sigma_1 \).
Again, the above definitions are legitimate, for, \( \nu \perp R(T) \) implies that \( \hat{\nu}(z_1) = 0 \) outside \( T \). This also shows that \( \sigma_1, \sigma_2 \) are supported on \( X_T \). To sum up, we have shown that for any \( \phi \in C_0^\infty(\mathbb{C}^2) \),
\[
\int \phi \, d\mu = \int \frac{\partial \phi}{\partial \overline{z}_1} \, d\sigma_1 + \int \frac{\partial \phi}{\partial \overline{z}_2} \, d\sigma_2
\]
where \( \sigma_i \)'s are supported on \( X_T \). Hence \( \mu \) is tame. \( \mu \) is not a zero measure because \( \int f(z_1) \, d\mu = \int f(z_1) \, dv \) for all \( f \in C(\mathbb{C}^1) \) and \( v \) is non-zero by hypothesis.

We note that if \( \mu \) is a tame measure on \( X \subset \mathbb{C}^n \), then \( \mathcal{F}_\mu \) is also tame for smooth function \( f \) with \( \partial f/\partial \overline{z}_j \equiv 0 \) on \( X \), \( i = 1, \ldots, n \). In particular, if \( \mu \) is as in (**) the measures \( z_2^m \mu, m = \pm 1, \pm 2, \ldots \), are all tame. Let \( S = \{ z_2^m \mu : \text{there is a nonzero orthogonal measure } \nu \text{ for } R(T) \text{ such that } \mu \text{ is defined by (**)}, m = 0, \pm 1, \pm 2, \ldots \} \).

We will show that

"If \( f \) in \( C(X_T) \) is such that \( f \) is annihilated by all elements in \( S \), then \( f \) is in \( R(X_T) \)."

Let \( \sum_{n=0}^{\infty} f_n(z_1) e^{int} \) be the "formal" Fourier expansion for \( \hat{f}(z_1, \theta) = f \circ p(z_1, \theta) = f(z_1, z_2) \) on \( \Gamma_{z_2} \). Let \( \sigma_j(z_1, z_2) = \sigma_j(z_1, \theta) \) be the \( j \)th Cesàro mean for \( f \). It is a straightforward generalization of Fourier series theory on the circle that \( \sigma_j \) converges uniformly to \( f \) on \( X_T \). So, in order to show \( f \in R(X_T) \), we need only to show \( \sigma_j \)'s in \( R(X_T) \) for all \( j \). Fix \( z_2^m \mu \) in \( S \),
\[
\int \sigma_j z_2^m \, d\mu = \int \left( \frac{1}{2\pi i} \int_{\Gamma_{z_2}} \sigma_j(z_1, z_2) z_2^m \, \frac{dz_2}{z_2} \right) \, dv(z_1)
\]
As \( j \to \infty \), \( \int \sigma_j z_2^m \, d\mu \to \int f z_2^m \, d\mu = 0 \) by hypothesis, while the right hand side approaches \( \int f_{-m}(z_1)(1 - z_1 \overline{z}_1)^{m/2} \, dv(z_1) \). So, we get \( \int f_{-m}(z_1)(1 - z_1 \overline{z}_1)^{m/2} \, dv(z_1) = 0 \) for all \( \nu \) in \( R(T)_+ \). Therefore, \( f_{-m}(z_1)(1 - z_1 \overline{z}_1)^{m/2} = h_{-m}(z_1) \) for some \( h_{-m}(z_1) \) in \( R(T) \). And
\[
\sigma_j(z_1, z_2) = \frac{1}{j} \sum_{n=0}^{j} \sum_{k=-n}^{n} f_k(z_1) e^{ik\theta}
\]
\[
= \frac{1}{j} \sum_{n=0}^{j} \sum_{k=-n}^{n} h_k(z_1)(1 - z_1 \overline{z}_1)^{k/2} e^{ik\theta}
\]
\[
= \frac{1}{j} \sum_{n=0}^{j} \sum_{k=-n}^{n} h_k(z_1) z_2^k
\]
is in \( R(X_T) \). So is \( f \).

We can now assert that the linear span of \( S \) is weak-* dense in \( R(X_T)_+ \). For, if not, then there exists \( g \in C(X_T) \) such that \( g \) annihilates \( S \) as well as its linear span,
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yet \( f g \, dt \neq 0 \) for some \( \tau \in R(X_T)^\perp \) which is not in the span of \( S \). By (\#), \( g \) is in \( R(X_T) \). Hence \( f g \, dt = 0 \), a contradiction. So the linear span of \( S \) is weak-* dense in \( R(X_T)^\perp \). Q.E.D.

Proof of Lemma.

\[
\frac{\partial \phi_0}{\partial \xi_1} = \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} \phi(\xi_1, t) \frac{dt}{2\pi} \right)
\]

\[
= \int_{-\pi}^{\pi} \left( \frac{\partial \phi}{\partial \xi_1} + \frac{\partial \phi}{\partial \xi_2} \cdot \frac{-\xi_1}{2(1 - \xi_1 \xi_2)^{1/2}} e^{it} + \frac{\partial \phi}{\partial \xi_2} \cdot \frac{-\xi_1}{2(1 - \xi_1 \xi_2)^{1/2}} e^{-it} \right) \frac{dt}{2\pi}.
\]

On the other hand,

\[
\frac{\partial \phi^*}{\partial t} = \frac{\partial \phi}{\partial \xi_2} \frac{\partial \xi_2}{\partial t} + \frac{\partial \phi}{\partial \xi_2} \frac{\partial \xi_2}{\partial t} = i \frac{\partial \phi}{\partial \xi_2} (1 - \xi_1 \bar{\xi}_1)^{1/2} e^{it} - i \frac{\partial \phi}{\partial \xi_2} (1 - \xi_1 \bar{\xi}_1)^{1/2} e^{-it}.
\]

So,

\[
\frac{\partial \phi}{\partial \xi_2} e^{it} = \frac{1}{i} \frac{\partial \phi^*}{\partial t} \frac{1}{(1 - \xi_1 \bar{\xi}_1)^{1/2}} + \frac{\partial \phi}{\partial \xi_2} e^{-it}.
\]

Substituting the above into (4), we get

\[
\frac{\partial \phi_0}{\partial \xi_1} = \int_{-\pi}^{\pi} \left( \frac{\partial \phi}{\partial \xi_1} + \frac{1}{2i} \frac{-\xi_1}{1 - \xi_1 \bar{\xi}_1} \frac{\partial \phi}{\partial \xi_2} + \frac{\partial \phi}{\partial \xi_2} \left( 1 - \xi_1 \bar{\xi}_1 \right)^{1/2} e^{-it} \right) \frac{dt}{2\pi}
\]

\[
= \int_{-\pi}^{\pi} \left( \frac{\partial \phi}{\partial \xi_1} + \frac{\partial \phi}{\partial \xi_2} \cdot \frac{-\xi_1}{\xi_2} \right) \frac{dt}{2\pi} = \left( \frac{\partial \phi}{\partial \xi_1} - \frac{\xi_1}{\xi_2} \frac{\partial \phi}{\partial \xi_2} \right)_0.
\]

The term \( j(\partial \phi^*/\partial t)(dt/2\pi) = 0 \), since, on \( \Gamma_{\xi_1} \), \( \partial \phi^*/\partial t = \Sigma \phi_{\alpha} e^{it} \) has no constant term. Q.E.D.

It is an immediate consequence of (\#) that we have

Theorem 2. Let \( \phi \) be a smooth function with \( \partial \phi/\partial z_i \equiv 0 \) on \( X_T \), \( i = 1, 2 \). Then \( \phi \in R(X_T) \).

Proof. Since \( \phi \) is annihilated by all elements of \( S \) so by (\#) \( \phi \in R(X_T) \). Q.E.D.

Next, we state a theorem about tame measures in general which is derived from the proof of a theorem due to Weinstock [2, Theorem 1.4].

Theorem (Weinstock). Let \( X \) be a compact subset of \( C^n \). If \( \mu \) is a tame measure on \( X \) and \( \{ U_a \}_{i=1}^N \) is a finite open covering of \( X \), then there exist \( \mu_a \) orthogonal measures for \( R(X \cap \overline{U_a}) \), where each \( \mu_a \) has its support contained in \( X \cap U_a \), and \( \mu = \Sigma \mu_a \).
PROOF. Let \( \{ \sigma_i \}_{i=1}^n \) be measures supported on \( X \), such that \( \mu = -\sum_{i=1}^n \frac{\partial \sigma_i}{\partial z_i} \). Let \( \{ \phi \alpha \} \) be a smooth partition of unity subordinate to \( \{ U_\alpha \} \) satisfying

(i) \( 0 < \phi_\alpha < 1 \) and \( \text{supp} \phi_\alpha \subset U_\alpha \),

(ii) \( \sum_1^N \phi_\alpha = 1 \).

Then,

\[
\mu = -\sum_{i=1}^n \frac{\partial}{\partial z_i} \left( \left( \sum_{\alpha=1}^N \phi_\alpha \right) \sigma_i \right) = -\sum_{i=1}^n \sum_{\alpha} \frac{\partial \phi_\alpha}{\partial z_i} \sigma_i - \sum_{i=1}^n \phi_\alpha \sum_{\alpha} \frac{\partial \sigma_i}{\partial z_i}.
\]

\[
= -\sum_{\alpha} \left( -\sum_{i=1}^n \frac{\partial \phi_\alpha}{\partial z_i} \sigma_i + \phi_\alpha \mu \right) = \sum_{\alpha} \mu_\alpha \text{, where } \mu_\alpha = -\sum_{i=1}^n \frac{\partial \phi_\alpha}{\partial z_i} \sigma_i + \phi_\alpha \mu.
\]

To show that \( \mu_\alpha \perp R(X \cap \bar{U_\alpha}) \), for any \( g \in R_0(X \cap \bar{U_\alpha}) \),

\[
\int g \, d\mu_\alpha = -\sum_{i=1}^n \int g \left( \frac{\partial \phi_\alpha}{\partial z_i} \right) d\sigma_i + \int g \phi_\alpha \, d\mu
\]

\[
= -\sum_{i=1}^n \int g \frac{\partial \phi_\alpha}{\partial z_i} \, d\sigma_i + \sum_{i=1}^n \int \frac{\partial (g \phi_\alpha)}{\partial z_i} \, d\sigma_i
\]

\[
= -\sum_{i=1}^n \int g \frac{\partial \phi_\alpha}{\partial z_i} \, d\sigma_i + \sum_{i=1}^n \int g \frac{\partial \phi_\alpha}{\partial z_i} \, d\sigma_i \text{ since } \frac{\partial g}{\partial z_i} = 0 \forall i
\]

\[
= 0.
\]

So \( \mu_\alpha \) annihilates \( R_0(X \cap \bar{U_\alpha}) \), hence will annihilate its closure \( R(X \cap \bar{U_\alpha}) \). Since \( \text{supp} \mu_\alpha \subset \text{supp} \phi_\alpha \cup \text{supp} \mu \); we have that \( \text{supp} \mu_\alpha \subset X \cap U_\alpha \) and the theorem is proved. Q.E.D.

With the help of (#) and the above theorem we can now assert that \( R(X_T) \) is a local algebra.

THEOREM 3. Let \( \{ U_\alpha \} \) be a finite open covering of \( X_T \). Let \( f \in C(X_T) \) be such that the restriction of \( f \) to \( X_T \cap \bar{U_\alpha} \) is in \( R(X_T \cap \bar{U_\alpha}) \) for all \( \alpha \). Then \( f \) is in \( R(X_T) \).

PROOF. Let \( S \) be as in Theorem 1. For any \( \mu \in S \), \( \mu \) is tame by Theorem 1. It follows from the above theorem that there exist \( \mu_\alpha \)'s such that \( \text{supp} \mu_\alpha \subset X_T \cap U_\alpha \), \( \mu_\alpha \perp R(X_T \cap \bar{U_\alpha}) \) and \( \sum \mu_\alpha = \mu \). Hence, by hypothesis,

\[
\int_{X_T} f \, d\mu = \sum_{\alpha} \int_{X_T \cap U_\alpha} f \, d\mu_\alpha = 0.
\]

\( f \) is annihilated by \( S \), and \( f \) is then in \( R(X_T) \) by (#). Q.E.D.
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References


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