PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 1, September 1980

ESSENTIALLY HERMITIAN OPERATORS IN $B(L_p)$

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ABSTRACT. It is shown that on $L_p[0, 1]$ all bounded linear operators which are Hermitian in the Calkin algebra $B(L_p)/C(L_p)$, must be of the form "Hermitian plus compact". That is, essentially Hermitian operators have the form, real multiplier plus compact.

1. Let X denote an infinite dimensional complex Banach space and $B(X)$ the corresponding space of bounded (resp. compact) linear operators on X. The Calkin algebra associated with X is given by $A(X) = B(X)/C(X)$. Many papers recently have dealt with variations of the following lifting question: Given that a coset $T + C(X)$ in $A(X)$ has a certain property, does the coset "lift" to an operator $T + K$, $K \in C(X)$, having the same property? For example, Stampfli **[8]** has shown that, if X is a separable complex Hilbert space, for every operator $T \in B(X)$ there is a compact operator K_T so that the Weyl spectrum of T and the spectrum of $T + K_T$ are equal.

In fact for most lifting theorems X is a separable infinite dimensional complex Hilbert space. Recently however, in an attempt to consider more general Banach spaces, these authors have proved that if $X = l_p$, then Hermitian elements in the Calkin algebra lift to the form "Hermitian plus compact". In this paper the above result is extended to the case $X = L_p[0, 1]$ (hereafter referred to as L_p): namely, the essentially Hermitian operators on $B(L_p)$, $1 < p < \infty$, $p \neq 2$, are of the form $M + K$ where M is a multiplication operator with an associated real-valued function in $L_\infty[0, 1]$ and $K \in C(L_p)$.

The conjecture that essentially Hermitian operators are of the form "real-valued multiplier plus compact" is the "natural" one due to Tam's result **[10]** that Hermitian operators on $B(X)$, X a function space not equivalent to L_2, are precisely the real-valued multiplier operators. In the case $p = 2$, the essentially Hermitian operators are exactly of the form "Hermitian plus compact". This follows from the easily derived fact that an operator on L_2 is essentially Hermitian if and only if its imaginary part is compact. As noted above the case $p \neq 2$ is much different.

Although some of the results of this paper are extensions of those in **[1]**, in most cases the techniques used are completely different. This is due to the fact that the structure of weakly null sequences in $L_p[0, 1]$ is more complex than the rather simple structure in l_p. Also the candidate for the real-valued multiplier part of an essentially Hermitian operator on $L_p[0, 1]$ is not so easily contrived as in the l_p case

Received by the editors April 20, 1979 and, in revised form, September 14, 1979.
1980 *Mathematics Subject Classification*. Primary 47B15; Secondary 47B38.

where the operators are matrices which have a multiplier part naturally associated with them, namely the diagonal of the matrix.

In the remainder of this section we state relevant definitions and results. Recall that both $B(X)$ and $A(X)$ are complex Banach algebras with unit. For a complex Banach algebra B with unit e and associated dual space B^*, define the state space S by

$$S \equiv \{f \in B^*: f(e) = 1 = \|f\|\}.$$

The *numerical range* of $x \in B$ is given as

$$W(x) \equiv \{f(x): f \in S\}.$$

It is known [9] that $W(x)$ is a nonempty, compact, convex subset of the complex plane and that the numerical radius $r(x) = \sup\{|z|: z \in W(x)\}$ is an equivalent norm on B. If $B = B(X)$ and $T \in B(X)$, the essential numerical range of T, $W_e(T)$ is defined to be the numerical range of $(T + C(X)) \in A(X)$. The essential norm of T satisfies the equation $\|T\|_e = \inf\{\|T + K\|: K \in C(X)\}$. An element $T \in B(X)$ is called Hermitian (resp. essentially Hermitian) if $W(T)$ (resp. $W_e(T)$) is an interval on the real axis. (For these definitions and more information on numerical ranges the reader is referred to the monographs of Bonsall and Duncan [2], [3].) Throughout the remainder of this paper the symbol $\mathfrak{D}$ will denote the subalgebra of $B(L_p)$ of bounded multiplier operators; that is, operators of the form $Tf = mf$ where $m \in L_\infty[0, 1]$.

In what follows, let

$$\mathfrak{P} = \{P: P \text{ is a projection onto a measurable subset of } [0, 1]\}.$$

If $P \in \mathfrak{P}$, $P^\perp$ denotes $I - P$.

For $T \in B(L_p)$, define the numerical radius

$$r(T) = \sup\{|s|: s \in W(T)\}$$

and the imaginary radius

$$r_i(T) = \sup\{|\operatorname{Im} s|: s \in W(T)\}.$$

As will be shown, $r_i(T)$ is a measure of the distance of T to the Hermitian operators in $B(L_p)$. We also define $r_e(T)$ and $r_{ei}(T)$ to be the analogous quantities for $W_e(T)$.

If ψ is any unit vector in L_p, then ψ' will denote a linear functional in the dual L_q of L_p such that $\psi'(\psi) = 1 = \|\psi'\|_q$. (If $p > 1$, ψ' is unique.) So given ψ, $\psi'(t) = \operatorname{sgn} \psi(t)|\psi(t)|^{p-1}$ where $\operatorname{sgn} \mu = e^{-i\theta}$ if $\mu = \rho e^{i\theta}$. Finally, if ϕ and ψ are unit vectors in L_p then

$$\psi'(\phi) = \langle \phi, \psi' \rangle = \int_0^1 \phi(t)\psi'(t)\, dt.$$

We begin with a result proved in [1, Lemma 1], for $T \in B(l_p)$. The proof carries over to $B(L_p)$ with only one minor change. Whereas in the proof of [1] there existed a projection $P \in \mathfrak{P}$ and unit vectors ϕ and ψ in L_p for which $\langle PTP^\perp\phi, \psi' \rangle = \sup_{P \in \mathfrak{P}} \|PTP^\perp\|$, due to the fact that T and either P or $P^\perp$ were compact operators, here, it may only be assumed that for given $\varepsilon > 0$, there exists a

projection $P \in \mathcal{P}$ and unit vectors ϕ and ψ satisfying $\langle PTP^{\perp}\phi, \psi' \rangle + \varepsilon \geqslant \sup_{P \in \mathcal{P}} \|PTP^{\perp}\|_p$. Nevertheless the following result still holds.

LEMMA 1. *Let $T \in B(L_p)$, $1 \leqslant p < \infty, p \neq 2$. Then*

$$\sup_{P \in \mathcal{P}} \|PTP^{\perp}\| \leqslant c_p r_i(T) < \infty$$

where

$$c_p^{-1} = \frac{1}{3p}\left((p-1)^{(p-1)/p} - (p-1)^{1/p}\right), \qquad p > 1,$$

and

$$c_p^{-1} = 1/3, \qquad p = 1.$$

In Lemma 2, below, the following observation is used: If ϕ and ψ are unit vectors in L_p having disjoint supports, and if c and d are complex scalars satisfying $|c|^p + |d|^p = 1$, then $c\phi + d\psi$ is a unit vector in L_p and

$$(c\phi + d\psi)' = \operatorname{sgn} c|c|^{p-1}\phi' + \operatorname{sgn} d|d|^{p-1}\psi'.$$

We also require the following fact: If $\{\sigma_n\}$ is a weakly null sequence of unit vectors then

$$r_{ei}(T) \geqslant \overline{\lim_{n\to\infty}} |\operatorname{Im}\langle T\sigma_n, \sigma_n' \rangle|.$$

This follows since a state ϕ on $A(L_p)$ having the form of a Banach limit may be constructed for which $|\operatorname{Im} \phi(T)| \geqslant \overline{\lim}_{n\to\infty}|\operatorname{Im}\langle T\sigma_n, \sigma_n' \rangle|$.

LEMMA 2. *Let $1 < p < \infty$, $p \neq 2$, and $T \in B(L_p)$. If T is essentially Hermitian and if $P, Q \in \mathcal{P}$ are disjoint (i.e. $PQ = 0$), then PTQ is compact.*

PROOF. Suppose that PTQ is not compact. Then there is a weakly null sequence $\{\phi_n\}$, $n = 1, 2, \ldots,$ of unit vectors for which $\lim\|PTQ\,\phi_n\| = a > 0$. Moreover the vectors ϕ_n can be chosen to have support in the measurable set associated with Q. Define $\psi_n^* = PTQ\phi_n$ and $\psi_n = \psi_n^*/\|\psi_n^*\|$, and set $\sigma_n = c\phi_n + d\psi_n$ so that $\|\sigma_n\| = |c|^p + |d|^p = 1$. Now

$$\begin{aligned}
r_{ei}(T) &\geqslant \overline{\lim_{n\to\infty}} |\operatorname{Im}\langle T\sigma_n, \sigma_n' \rangle| \\
&= \overline{\lim}\, |\operatorname{Im}\{\langle PTP\sigma_n, \sigma_n' \rangle + \langle PTQ\sigma_n, \sigma_n' \rangle + \langle QTP\sigma_n, \sigma_n' \rangle + \langle QTQ\sigma_n, \sigma_n' \rangle\}| \\
&= \overline{\lim}\, |\operatorname{Im}\{\langle PTPd\psi_n, \operatorname{sgn} d|d|^{p-1}\psi_n' \rangle + \langle PTQc\phi_n, \operatorname{sgn} d|d|^{p-1}\psi_n' \rangle \\
&\qquad + \langle QTPd\psi_n, \operatorname{sgn} c|c|^{p-1}\phi_n' \rangle + \langle QTQc\phi_n, \operatorname{sgn} c|c|^{p-1}\phi_n' \rangle\}| \\
&\geqslant \overline{\lim}\, |\operatorname{Im}\{\langle PTQc\phi_n, \operatorname{sgn} d|d|^{p-1}\psi_n' \rangle + \langle QTPd\psi_n, \operatorname{sgn} c|c|^{p-1}\phi_n' \rangle\}| \\
&\quad - 2r_{ei}(T).
\end{aligned}$$

So

$$r_{ei}(T) \geqslant \overline{\lim}\, \frac{1}{3}\left|\operatorname{Im}\{c \operatorname{sgn} d|d|^{p-1}\langle PTQ\phi_n, \psi_n' \rangle + d \operatorname{sgn} c|c|^{p-1}\langle QTP\psi_n, \phi_n' \rangle\}\right|.$$

By taking subsequences of ϕ_n if necessary we may assume either

(i) $\lim|\langle QTP\psi_n, \phi_n'\rangle| \equiv b \leqslant a$ and $\lim\langle PTQ\phi_n, \psi_n'\rangle = a$, or

(ii) $\lim|\langle QTP\psi_n, \phi_n'\rangle| \equiv b \geqslant a$ and $\lim\langle PTQ\phi_n, \psi_n'\rangle = a$.

We proceed assuming condition (i). Select c and d to make $c \operatorname{sgn} d$ purely imaginary. The maximum of $|c|\,|d|^{p-1}$, for $|c|^p + |d|^p = 1$, occurs at $d = ((p-1)/p)^{1/p}$. Thus

$$r_{ei}(T) \geqslant \frac{1}{3}\left|\left(\frac{1}{p}\right)^{1/p}\left(\frac{p-1}{p}\right)^{(p-1)/p} - \left(\frac{1}{p}\right)^{(p-1)/p}\left(\frac{p-1}{p}\right)^{1/p}\right| a = ac_p^{-1}.$$

This contradicts the hypothesis that T is essentially Hermitian, and so PTQ is compact. In the case of (ii), select yet another subsequence of $\{\phi_n\}$ so that $\lim\langle QTP\psi_n, \phi_n'\rangle = |b|e^{i\theta}$ for some fixed angle θ. Next select c and d to make $d \operatorname{sgn} c \lim_{n\to\infty}\langle QTP\psi_n, \phi_n'\rangle$ purely imaginary. It follows, as in (i), that PTQ must be compact.

Lemma 2 is really analogous to Lemma 1 but with its setting in the Calkin algebra $\mathcal{Q}(L_p)$. It asserts that if T is Hermitian in $\mathcal{Q}(L_p)$ and P, Q are disjoint projections in $\mathcal{P}$ then $PTQ = 0$ in $\mathcal{Q}(L_p)$. That is, essentially Hermitian operators are "diagonal" in $\mathcal{Q}(L_p)$ with respect to $\mathcal{P}$.

Recall that the multiplier operators $\mathcal{D}$ viewed as a subspace of $B(L_p)$ for any $1 \leqslant p \leqslant \infty$ is isometrically isomorphic to L_∞. Thus, $\mathcal{D}$ is a C^*-algebra with the $*$ operation being complex conjugation. Let $\mathcal{D}^u$ denote the set of multipliers u for which $u^* \cdot u = \bar{u} \cdot u = 1$. For $T \in B(L_p)$, define $K(T)$ to be the weak operator closure of the convex hull of the set $\{u^*Tu\colon u \in \mathcal{D}^u\}$. From [**4**, Problem 6, p. 512], $K(T)$ is seen to be weak operator compact for $p > 1$. The next lemma asserts that $K(T) \cap \mathcal{D} \neq \varnothing$, and was motivated by the heuristic argument following Lemma 1.4 in [**6**].

LEMMA 3. *Let $T \in B(L_p)$, $p > 1$. Then $K(T) \cap \mathcal{D} \neq \varnothing$.*

PROOF. First note that $K(T)$ is convex and compact in the weak operator topology. Let $F = \{$all mappings in $B(L_p)$ of the type $T_u(B) = u^*Bu | u \in \mathcal{D}^u\}$. It is clear that F is a self-commuting family of mappings and that $K(T)$ is invariant under F. An application of the Markov-Kakutani theorem [**4**, p. 456] yields an $M \in K(T)$ such that $T_u(M) = M$ for all $T_u \in F$. It remains to show that $M \in \mathcal{D}$. To see this note that $T_u(M) = M$ for all $T_u \in F$ means that $u^*Mu = M$ for all $u \in \mathcal{D}^u$, or equivalently $Mu = uM$. Let $f_n(t) = e^{i2\pi nt} \in L_p$. Then $M(u(f_n)) = u(M(f_n))$. Letting $u = e^{-i2\pi nt}$, it follows that $M(1) = e^{-i2\pi nt}M(e^{i2\pi nt})$, or $M(e^{i2\pi nt}) = M(1)e^{i2\pi nt}$. This implies that M is multiplication by $M(1)$ on the closure of the trigonometric polynomials, that is, on $L_p[0, 1]$.

With this lemma it is possible to establish the results analogous to Lemmas 5 and 6 of [**1**].

DEFINITION. For $T \in B(L_p)$, define

$$\|D_T|\mathcal{D}\| = \sup\{\|ST - TS\|\colon S \in \mathcal{D}, \|S\| = 1\}.$$

LEMMA 4. *For $T \in B(L_p)$,*

$$\operatorname{dist}(T, \mathcal{D}) \leqslant \|D_T|\mathcal{D}\| \leqslant 8 \sup_{P \in \mathcal{P}} \|PTP^\perp\|.$$

REMARK. The proof of Lemma 4 is nearly identical to the proofs of Lemmas 5 and 6 of [1] and will be omitted. It should be remarked that any element in $K(T) \cap \mathfrak{D}$ may be used to replace the operator diag A in Lemma 5 of [1].

The way is now paved for

THEOREM 5. *Let* $1 < p < \infty$, $p \neq 2$, *and* $T \in B(L_p)$. *Then* T *is essentially Hermitian iff* T *has the form real multiplier plus compact.*

PROOF. The sufficiency is easy and follows from [3, p. 127] so it remains to establish the necessity. Suppose T is essentially Hermitian. We proceed to construct a real-valued multiplier operator which is a compact perturbation of T. Partition the interval [0, 1] into 2^n equal subintervals $\Delta_{n,i}$, $i = 1, 2, \ldots, 2^n$, and let $P_{n,i}$ be the projection (operator) associated with $\Delta_{n,i}$. By Lemma 2 it follows that $P_{n,i}TP_{n,j}$ is compact if $i \neq j$. Hence $\sum_{i=1}^{2^n} P_{n,i}TP_{n,i} - T$ is compact for each n.

CLAIM. $\lim_{n\to\infty} \sup_{1\leqslant i\leqslant 2^n} \operatorname{dist}(P_{n,i}TP_{n,i}, \mathfrak{D}) = 0$.

PROOF OF CLAIM. The argument proceeds by contradiction. By taking subsequences if necessary, assume that for each n there exists $1 \leqslant m \leqslant 2^n$ for which $\operatorname{dist}(P_{n,m}TP_{n,m}, \mathfrak{D}) \geqslant \varepsilon$ for some $\varepsilon > 0$. By Lemma 4, it follows that there are projections $Q_{n,m}$ and $Q_{n,m}^{\perp}$ having support in $\Delta_{n,m}$ for which $\|Q_{n,m}TQ_{n,m}^{\perp}\| \geqslant \varepsilon/8$. Lemma 1 assures that $r_i(P_{n,m}TP_{n,m}) \geqslant c_p^{-1}\varepsilon/8$ and so one may select vectors v_n so that $\operatorname{Im}\langle P_{n,m}TP_{n,m}v_n, v_n'\rangle \geqslant c_p^{-1}\varepsilon/8$. Since the measure of $\Delta_{n,m} \to 0$ as $n \to \infty$, a subsequence of the $\{v_n\}$ (call it $\{v_n\}$ also) converges weakly to zero and $\operatorname{Im}\langle Tv_n, v_n'\rangle = \operatorname{Im}\langle P_{n,m}TP_{n,m}v_n, v_n'\rangle \geqslant c_p^{-1}\varepsilon/8$. Thus $r_{ei}(T) \geqslant c_p^{-1}\varepsilon/8$. Since this contradicts the fact that T is essentially Hermitian, the claim is proved.

Since $\operatorname{dist}(\sum_i^{2^n} P_{n,i}TP_{n,i}, \mathfrak{D}) = \sup_{1\leqslant i\leqslant 2^n} \operatorname{dist}(P_{n,i}TP_{n,i}, \mathfrak{D})$ and this latter quantity tends to zero as n increases to infinity it follows that there exist real-valued multipliers D_n for which

$$\left\| D_n - \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} \right\| \to 0 \quad \text{as } n \to \infty.$$

We claim that the D_n form a Cauchy sequence in the uniform operator topology. Indeed, since for any $M \in \mathfrak{D}$, $\|M\| = \|M\|_e$,

$$\begin{aligned}
\|D_n - D_m\| &= \|D_n - D_m\|_e \\
&\leqslant \Bigg\| D_n - \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} + \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} - T + T \\
&\qquad - \sum_{i=1}^{2^m} P_{m,i}TP_{m,i} + \sum_{i=1}^{2^m} P_{m,i}TP_{m,i} - D_m \Bigg\|_e \\
&\leqslant \left\| D_n - \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} \right\|_e + \left\| \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} - T \right\|_e \\
&\quad + \left\| \sum_{i=1}^{2^m} P_{m,i}TP_{m,i} - T \right\|_e + \left\| \sum_{i=1}^{2^m} P_{m,i}TP_{m,i} - D_m \right\|_e \\
&= \left\| D_n - \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} \right\|_e + \left\| \sum_{i=1}^{2^m} P_{m,i}TP_{m,i} - D_m \right\|_e \\
&\leqslant \left\| D_n - \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} \right\| + \left\| \sum_{i=1}^{2^m} P_{m,i}TP_{m,i} - D_m \right\|
\end{aligned}$$

which by the claim proved above is small for m and n sufficiently large. Thus $\{D_n\}$ is Cauchy and there is a multiplier D for which $\lim D_n = D$. Now for any $\varepsilon > 0$, and for n sufficiently large

$$\begin{aligned}\|T - D\|_e &= \left\| T - \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} + \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} - D_n + D_n - D \right\|_e \\ &\leqslant \left\| T - \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} \right\|_e + \left\| \sum_{i=1}^{2^n} P_{n,i}TP_{n,i} - D_n \right\|_e + \|D_n - D\|_e \\ &\leqslant 0 + \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Since ε is arbitrary $\|T - D\|_e = 0$ and so $T = D + K$ with K compact. This completes the proof.

2. Essentially Hermitian operators are essential multipliers. In this section we derive a proof that essentially Hermitian operators are essential multipliers (i.e. operators T for which $TM - MT$ is compact for all multipliers M) *without* using the results of the previous section. It seems possible to us this approach might be useful in characterizing essentially Hermitian elements in other spaces.

PROPOSITION 6. *Let $1 \leqslant p < \infty$, $p \neq 2$. If $T \in B(L_p)$ is essentially Hermitian then T is an essential multiplier.*

PROOF. Let $M \in \mathfrak{D}$. Then for every $\varepsilon > 0$ there exists a simple function $M_\varepsilon \in L_\infty[0, 1]$ such that $\|M - M_\varepsilon\|_\infty < \varepsilon$. If $M_\varepsilon = \sum_{i=1}^n \alpha_i\chi_i$ with χ_i a characteristic function, let P_i be the projection onto the support of χ_i. Since $P_iP_j = 0$, $i \neq j$, P_iTP_j is compact by Lemma 2. The operator

$$M_\varepsilon T - TM_\varepsilon = \sum_{i \neq j} (\alpha_i - \alpha_j)P_iTP_j$$

is also compact. Let M_n be a sequence of simple functions satisfying $\|M_n - M\|_\infty < n^{-1}$. Then

$$MT - TM = (M - M_n)T - T(M - M_n) + M_nT - TM_n.$$

Clearly $(M - M_n)T$ and $T(M - M_n)$ converge to zero in the uniform operator topology. Thus $M_nT - TM_n$ converges to $MT - TM$ in the same topology. But $M_nT - TM_n$ is compact; therefore $MT - TM$ is also compact. This completes the proof.

Johnson and Parrott have shown [**6**] that if $\mathcal{A}$ is a commutative von Neumann algebra and if T essentially commutes with $\mathcal{A}$ (i.e. $TA - AT$ is compact for all $A \in \mathcal{A}$), then T is a compact perturbation of an element of the commutant of A. It seems likely that Theorem 2.1 of [**6**] could be adapted to our situation to conclude that essentially Hermitian operators (which are essential multipliers by Proposition 6) must be of the form multiplier plus compact and thus give another proof of Theorem 5. The modification is not immediate however since in [**6**] the fact that $L(H)$ is a dual space is used quite extensively. Moreover our approach illustrates the fact that $r_{ei}(T)$ "measures" the distance of T to the essentially Hermitian operators. In general $r_i(T)$ is *not* equivalent to the distance to the Hermitians as shown by an example in [**1**].

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