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TAYLOR-DIRICHLET SERIES AND ALGEBRAIC DIFFERENTIAL-DIFFERENCE EQUATIONS[1]

FRANK WADLEIGH

ABSTRACT. It is proved that if a convergent Taylor-Dirichlet series

$$\sum_{k=0}^{\infty} P_k(s)e^{-\lambda_k s}, \qquad s = \sigma + it, \lambda_k \in \mathbf{C}, P_k(s) \in \mathbf{C}[s], \Re(\lambda_k)\uparrow\infty,$$

satisfies an algebraic differential-difference equation then the set of its exponents $\{\lambda_k\}_{k=0}^{\infty}$ has a finite, linear, integral basis. This generalizes a theorem of A. Ostrowski.

An application of the theorem to a problem of oscillation in neuro-muscular systems is indicated.

1. Introduction. According to a well-known theorem of A. Ostrowski [7, Satz **6**, p. 260] if a convergent Dirichlet series

$$\sum_{k=0}^{\infty} a_k e^{-\lambda_k s}, \qquad s = \sigma + it, a_k \in \mathbf{C},$$

with real exponents $\lambda_0 < \lambda_1 < \dots \to \infty$ satisfies an algebraic differential-difference equation then the set of its exponents $\{\lambda_k\}_{k=0}^{\infty}$ has a finite, linear, integral basis.

In this paper we show that Ostrowski's theorem continues to hold for a convergent Taylor-Dirichlet series

$$\phi(s) = \sum_{k=0}^{\infty} P_k(s)e^{-\lambda_k s}, \qquad \lambda_k \in \mathbf{C}, P_k(s) \not\equiv 0, \tag{1}$$

where the P_k are polynomials with complex coefficients and the real parts λ_k' of λ_k satisfy $\lambda_0' < \lambda_1' < \dots \to \infty$. The theorem is stated and proved in §3. In §4 we briefly indicate an area of application to neuro-muscular systems.

We recall that an algebraic differential-difference equation has the form

$$G(x, f^{(m_1)}(x + h_1), \dots, f^{(m_r)}(x + h_r)) = 0 \tag{2}$$

where G is a polynomial

$$G(x, x_1, \dots, x_r) = \sum C_{k_1, \dots, k_r} x_1^{k_1} \dots x_r^{k_r}$$

with coefficients depending on x, and the ordered pairs $(h_1, m_1), \dots, (h_r, m_r)$ of real numbers h_k and nonnegative integers m_k are distinct. By the total degree of G is meant, as usual, the maximum value of the sums $k_1 + \dots + k_r$. For a discussion

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of convergence of (1) refer to Valiron [10] where it is shown that if the degrees μ_k of $P_k(s)$ and the exponents λ_k satisfy the conditions

$$\lim_{k\to\infty} \frac{\mu_k}{\lambda_k} = 0, \qquad \lim_{k\to\infty} \frac{\log k}{\lambda_k} = 0$$

then the region of convergence of (1) is the same as that of the associated classical Dirichlet series $\sum_1^\infty A_k \exp(-s\lambda_k)$, where A_k is the maximum of the moduli of the coefficients of P_k. Convergence of these latter series has been treated by a number of authors, for instance, Väisälä [9] and Miškelevičius [5]. Further results on Taylor-Dirichlet series can be found in Lepson [3], Lunc [4] and Blambert and Berland [1].

2. Lemmas. The proof of Ostrowski's theorem depends on three lemmas, the first of which [7, p. 246] states that an analytic function $g(s)$ which satisfies an algebraic differential-difference equation of the form (2) also satisfies an equation

$$F\big(f^{(m_1)}(s+h_1), \ldots, f^{(m_r)}(s+h_r)\big) = 0 \tag{3}$$

where the polynomial F does not contain the variable s as one of its arguments and where the total degree of F does not exceed that of G. This result carries over without change to (1). Lemmas 1 and 2 which follow are modifications, respectively, of Ostrowski's Hilfssatz 2, p. 247 and Hilfssatz 3, p. 248 of [7].

LEMMA 1. *The exponential polynomial*

$$E(\lambda) = \sum_{k=1}^{q} P_k(\lambda)e^{\alpha_k\lambda} \not\equiv 0, \qquad P_k(\lambda) \in \mathbf{C}[\lambda],$$

where the exponents α_k are distinct real numbers has some zero-free right half-plane; i.e. there exists a positive number B such that all zeros of $E(\lambda)$ have real part less than B.

PROOF. Well known; see, e.g., Langer [2, p. 224].

DEFINITION 1. A Taylor-Dirichlet series (1) is said to satisfy the algebraic differential-difference equation (3) *formally* if all coefficients of the series arising from formal substitution of (1) in the left-hand side of (3) vanish identically.

DEFINITION 2. Throughout, the 'exponent' of a term of the form $P(s)e^{-\lambda s}$ will refer to the number λ.

We now state and prove the main lemma.

LEMMA 2. *If the Taylor-Dirichlet series* (1) *satisfies an algebraic differential-difference equation formally in which the variable s does not occur explicitly then every exponent λ_k of sufficiently large index k can be expressed as a linear, integral combination of the preceding exponents $\lambda_0, \lambda_1, \ldots, \lambda_{k-1}$.*

PROOF. We prove the lemma for $0 \leqslant \lambda_0'$ since the general situation is treated exactly as in [7]. By hypothesis, $\phi(s)$ satisfies an algebraic differential-difference equation of the form (3) formally. Let N be the total degree of the polynomial F and suppose that $\phi(s)$ does not satisfy formally any nontrivial algebraic differential-difference equation of total degree less than N.

The partial derivatives

$$F_\rho(f^{(m_\nu)}(s + h_\nu)) = \frac{\partial F(f^{(m_\nu)}(s + h_\nu))}{\partial(f^{(m_\rho)}(s + h_\rho))}$$

where we have written $F(f^{(m_\nu)}(s + h_\nu))$ in place of $F(f^{(m_1)}(s + h_1), \dots, f^{(m_r)}(s + h_r))$ are clearly polynomials in $f^{(m_\nu)}(s + h_\nu)$, $\nu = 1, 2, \dots, r$, of total degree less than N.

By the minimality of N, the expression $F_\rho(\phi^{(m_\nu)}(s + h_\nu))$ does not vanish formally. We can write the first (ordered by real parts of exponents) nonzero term of this series in the form

$$Q_\rho(s)e^{-\Lambda_\rho s} \tag{4}$$

where Λ_ρ is a linear, integral combination of a finite number of λ_k's and $Q_\rho(s)$ is a polynomial in s with coefficients depending on these same λ_k's. Let $l_0 = \min_{1 \leq \rho \leq r}\{\Lambda'_\rho\}$ and suppose there are m such terms so that $\Lambda'_{\rho_1} = \cdots = \Lambda'_{\rho_m} = l_0$ and that for all other terms of the form (4) we have $\Lambda'_\rho > l_0$. Clearly only a finite number of terms of the series $\phi^{(m_\nu)}(s + h_\nu)$ contribute to the leading nonzero term (4) (or to the preceding terms which have cancelled). It follows from this and the monotonicity of the λ'_k that a positive number l_1 exists such that if

$$\lambda'_n > l_1 \tag{5}$$

then $\phi(s)$ can be split up into two sums

$$\phi(s) = A_n(s) + R_n(s)$$

where

$$A_n(s) = \sum_{k=0}^{n-1} P_k(s)e^{-\lambda_k s}, \qquad R_n(s) = \sum_{k=n}^{\infty} P_k(s)e^{-\lambda_k s}$$

and the series $F_\rho(A_n^{(m_\nu)}(s + h_\nu))$ will have the same leading nonzero term as $F_\rho(\phi^{(m_\nu)}(s + h_\nu))$, namely (4).

Application of Taylor's formula for multivariate polynomials yields the finite expansion

$$\begin{aligned} F(\phi^{(m_\nu)}(s + h_\nu)) = {} & F(A_n^{(m_\nu)}(s + h_\nu)) + \sum_{1 \leq \rho \leq r} F_\rho(A_n^{(m_\nu)}(s + h_\nu))R_n^{(m_\rho)}(s + h_\rho) \\ & + \sum_{1 \leq \rho \leq \eta \leq r} F_{\rho,\eta}(A_n^{(m_\nu)}(s + h_\nu))R_n^{(m_\rho)}(s + h_\rho)R_n^{(m_\eta)}(s + h_\eta) \\ & + \dots \end{aligned} \tag{6}$$

where $F_{\rho,\eta}$ denotes the second-order derivative apart from factorial coefficients.

Consider the third member on the right-hand side of (6) (reserving the word 'term' for series). A lower bound for the real part of the exponent of the first nonzero term of this expression as a formal Taylor-Dirichlet series is clearly $2\lambda'_n$, with corresponding lower bounds for the fourth, fifth, ... members on the right-hand side of (6) being $3\lambda'_n$, $4\lambda'_n$, etc.

In order to express $R_n^{(m_\rho)}(s)$ in a simple form we use the notation $D = d/ds$, the formal differential operator:

$$R_n^{(m_\rho)}(s) = \frac{d^{(m_\rho)}}{ds^{m_\rho}}\left(\sum_{k=n}^{\infty} P_k(s)e^{-\lambda_k s}\right) = e^{-\lambda_n s}(D - \lambda_n)^{m_\rho}(P_n(s)) + \cdots.$$

We can now write the initial term of the second member on the right-hand side of (6) as

$$e^{-(l_0+\lambda_n)}E(s, \lambda_n), \tag{7}$$

where

$$E(s, \lambda) = \sum_{j=1}^{m} Q_{\rho_j}(s)(D - \lambda)^{m_\rho}(P_n(s))e^{-\lambda h_{\rho_j}}$$

and is, for fixed s, an exponential polynomial of the kind considered in Lemma 1. Since the polynomials $Q_{\rho_1}, \ldots, Q_{\rho_r}$ and P_n are $\not\equiv 0$ and the ordered pairs $(h_{\rho_1}, m_{\rho_1}), \ldots, (h_{\rho_r}, m_{\rho_r})$ are distinct, there exists (at least) one value of s, say $s = s^*$ for which $E(s^*, \lambda) \not\equiv 0$ as a function of λ.

We may therefore apply Lemma 1 to $E(s^*, \lambda)$ and infer the existence of a positive number l_2 such that for all λ_n whose real part λ_n' satisfies

$$\lambda_n' > l_2 \tag{8}$$

we have $E(s^*, \lambda_n) \neq 0$.

The term (7) therefore cannot cancel with the initial term of the third, fourth, ... members on the right-hand side of (6) provided $l_0 + \lambda_n' < 2\lambda_n'$; that is, if

$$\lambda_n' > l_0. \tag{9}$$

Combining the above we see from (5), (8) and (9) that if we take $\lambda_n' > l_0 + l_1 + l_2$ then (7) holds and the nonvanishing term (4) cannot be cancelled by any of the third, fourth, ... members on the right-hand side of (6). By hypothesis, however, each term of the series on the right-hand side of (6) must vanish; therefore $F(A_n^{(m_\nu)}(s + h_\nu))$ must contain a term whose exponent is $l_0 + \lambda_n$. Hence by the definition of $A_n(s)$, $l_0 + \lambda_n$ must be a linear, integral combination of $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$. The same is true for l_0 since $\lambda_n' > l_0$, and hence we see that λ_n itself is a linear, integral combination of the preceding exponents.

3. Statement and proof of the theorem.

THEOREM. *If the convergent Taylor-Dirichlet series* (1) *satisfies an algebraic differential-difference equation then the set of its exponents* $\{\lambda_k\}_{k=0}^{\infty}$ *has a finite, linear, integral basis.*

PROOF. By the remarks at the beginning of §2 it suffices to show that if $\phi(s)$ (defined by (1)) satisfies (3) then it also satisfies this equation formally, for then we may use Lemma 2 to complete the proof. In order to obtain the desired contradiction we therefore assume that $\phi(s)$ does *not* satisfy (3) formally. We can write the first (ordered, as before, according to real parts of exponents) nonzero term of the series obtained by formal substitution of $\phi(s + h_\nu)$ into the left-hand side of (3) as

$$Q(s)e^{-\Lambda s} \tag{10}$$

where Λ is a linear combination of a finite number of λ_k's and $Q(s)$ is a polynomial in s whose coefficients depend on these λ_k's. We obviously have

$$Q(s) \not\equiv 0. \tag{11}$$

We proceed to split $\phi(s)$ into two sums as in the proof of Lemma 2, $\phi(s) = A_n(s) + R_n(s)$ where n is taken so large that $\lambda_n' > \Lambda'$, thus ensuring that the first nonzero term of $F(A_n^{(m_\nu)}(s + h_\nu))$ will also be $Q(s)e^{-\Lambda s}$.

Writing

$$S_n(s) = P_n(s) + P_{n+1}(s)e^{-(\lambda_{n+1}-\lambda_n)s} + \ldots$$

then

$$\phi(s) = A_n(s) + e^{-\lambda_n s}S_n(s) \tag{12}$$

and

$$S_n(s) = O(s^{\mu_n}), \qquad s \to \infty,$$

where $\mu_n = \deg P_n(s)$.

Here and subsequently all limits are taken as s assumes real values. From (12) we get

$$\phi^{(m_\nu)}(s + h_\nu) = A_n^{(m_\nu)}(s + h_\nu) + e^{-\lambda_n s}T_n^{(m_\nu)}(s, \lambda_n) \tag{13}$$

where

$$T_n^{(m_\nu)}(s, \lambda_n) = O(s^{\mu_n}), \qquad s \to \infty. \tag{14}$$

Replacing (12) and (13) into the left-hand side of (3) and applying Taylor's formula we have

$$0 = F\big(\phi^{(m_\nu)}(s + h_\nu)\big) = F\big(A_n^{(m_\nu)}(s + h_\nu)\big) + e^{-\lambda_n s}\Phi(s), \tag{15}$$

where

$$\Phi(s) = O(s^M), \qquad s \to \infty, \tag{16}$$

for some positive integer M.

On the other hand we can write

$$F\big(A_n^{(m_\nu)}(s + h_\nu)\big) = e^{-\Lambda s}(Q(s) + h(s)) \tag{17}$$

with $h(s) \to 0$ as $s \to \infty$.

Thus if we insert the right-hand side of (17) in (15) we get

$$0 = e^{-\Lambda s}(Q(s) + h(s)) + e^{-\lambda_n s}\Phi(s). \tag{18}$$

Multiplying (18) by $e^{\Lambda s}$ and taking the limit as $s \to \infty$ we obtain $Q(s) \equiv 0$ which contradicts (11) and proves the theorem.

4. An application. We indicate briefly how our theorem may be applied to a solution of the following algebraic differential-difference equation

$$\sum_{k=0}^{4} a_k f^{(k)}(t) + \sum_{k=1}^{3} \left\{ b_k f(t - t_k) + c_k f^{(1)}(t - t_k) \right\} = 0 \tag{19}$$

occurring in the investigation of oscillations in neuro-muscular systems [6]. Here we write $f^{(0)}(t)$ for $f(t)$ and $f^{(1)}(t)$ for $f'(t)$. The a_k, b_k, c_k denote complex constants and $0 < t_1 < t_2 < t_3$.

Substituting $f(t) = e^{st}$ in (19) gives

$$e^{t_3 s} \sum_{k=0}^{4} a_k s^k + \sum_{k=1}^{3} (b_k + c_k s) e^{(t_3 - t_k)s} = 0 \tag{20}$$

for the transcendental characteristic equation. By a well-known theorem of Pontryagin [8] (20) has in general infinitely many roots s with negative real part. There are, moreover, only finitely many multiple roots.

Denote the roots of (20) having negative real part by

$$s_k = -p_k + iq_k, \qquad p_k > 0,\ k = 1, 2, \ldots,$$

where s_k has multiplicity n_k.

The corresponding particular solutions of (19) are then

$$t^\nu e^{-p_k t} \cos q_k t, \qquad t^\nu e^{-p_k t} \sin q_k t, \qquad \nu = 0, 1, \ldots, n_k - 1.$$

The general solution of (20) has the form

$$f(t) = \sum_{k=1}^{\infty} e^{-p_k t} \{ P_k(t) \cos q_k t + Q_k(t) \sin q_k t \} \tag{21}$$

where $P_k(t)$ and $Q_k(t)$ are polynomials in t of degree $n_k - 1$ with complex coefficients such that (21) converges absolutely and uniformly for $t \geqslant 0$ and is four times differentiable.

Applying our theorem to the function $f(t)$ there exists then a finite set of numbers $p_1, p_2, \ldots, p_n$ such that

$$f(t) = \sum_{k=1}^{\infty} e^{-\Lambda_k t} \{ P_k(t) \cos q_k t + Q_k(t) \sin q_k t \}$$

where $\Lambda_k = \sum_{j=1}^{n} \alpha_{kj} p_j$, for some integers α_{kj}.

The application described above has served primarily as an example to illustrate the possible usefulness of the theorem. It is intended to pursue further such applications in a subsequent paper.

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NATIONAL RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, PRETORIA, REPUBLIC OF SOUTH AFRICA

Current address: Department of Bioengineering, University of California at San Diego, La Jolla, California 92093