CYCLIC VECTORS OF LAMBERT'S WEIGHTED SHIFTS

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ABSTRACT. Let \( B(H) \) denote the Banach algebra of all bounded linear operators on an infinite-dimensional separable complex Hilbert space \( H \), and let \( l^2 \) be the Hilbert space of all square-summable complex sequences \( x = \{x_0, x_1, x_2, \ldots \} \). For an injective operator \( A \) in \( B(H) \) and a nonzero vector \( f \) in \( H \), put
\[
w_m = \frac{\|A^mf\|}{\|A^{m-1}f\|}, \quad m = 1, 2, \ldots
\]
The operator \( T_{Af} \) on \( l^2 \), defined by
\[
T_{Af}(x) = \{ \frac{w_1x_1, w_2x_2, \ldots} {w_0, w_1, w_2, \ldots} \},
\]
is called a weighted (backward) shift with the weight sequence \( \{w_m\}_{m=1}^{\infty} \). This paper is concerned with the investigation of the existence of cyclic vectors of \( T_{Af} \). Also it is shown that if \( A \) satisfies certain nice conditions, then every transitive subalgebra of \( B(H) \) containing \( T_{Af} \) coincides with \( B(H) \).

1. Let \( H \) be an infinite-dimensional separable complex Hilbert space with an orthonormal basis \( \{e_m\}_{m=0}^{\infty} \) and let \( B(H) \) be the Banach algebra of all (bounded linear) operators from \( H \) into \( H \). If \( \{w_m\}_{m=1}^{\infty} \) is a bounded sequence of nonzero complex numbers, the unique operator \( T \) on \( H \) defined by \( Te_0 = 0 \) and \( Te_m = w_m e_{m-1} \), \( m = 1, 2, \ldots \), is called the (unilateral backward) weighted shift with the weight sequence \( \{w_m\}_{m=1}^{\infty} \). A concrete realisation of the weighted shift is obtained by considering the Hilbert space \( l^2 \) of all square-summable complex sequences \( x = \{x_0, x_1, x_2, \ldots \} \). The weighted shift \( T \) on \( l^2 \) appears as
\[
T(x_0, x_1, x_2, \ldots) = (w_1x_1, w_2x_2, \ldots).
\]
A vector \( x_0 \in H \) is called a cyclic vector of \( T \) if \( \bigvee_{n=0}^{\infty} (T^nx_0) = H \). The existence of cyclic vectors of weighted shifts has been the subject of investigation by many authors; see, for example, Douglas, Shapiro and Shields [3], [4], Gellar [6], Herrero [7], Deddens, Gellar and Herrero [2], Nikolskiï [11], [12] and Rabindranathan [14].

We say that an operator \( A \) is power-bounded if \( \|A^n\| < \delta \) for all \( n = 1, 2, 3, \ldots \), where \( \delta \) is a constant.

2. Let \( A \in B(H) \) and suppose that \( A \) is injective. For a vector \( f \neq 0 \) in \( H \), let \( T_{Af} \) be the weighted shift on \( l^2 \) with the weight sequence \( \{w_m\}_{m=1}^{\infty} \), where
\[
w_m = \frac{\|A^mf\|}{\|A^{m-1}f\|}.
\]

The subnormality of the weighted shifts \( T_{Af} \) has been recently studied by Lambert [10]. In this paper we exhibit the existence of cyclic vectors of such weighted shifts under nice conditions on the operator \( A \).

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We set
\[ \Delta_k = \sum_{m>k} \frac{x_m^2}{x_k^2}, \quad k = 0, 1, 2, \ldots, \] (2)
and prove the following

**Theorem 1.** Let \( A \) be power-bounded and such that for every nonzero vector \( f \in H, A^nf \) does not converge to 0 as \( n \to \infty \). Then any vector \( x = \{x_m\}_{m=0}^\infty \) in \( l^2 \), such that

(i) \( x_m \neq 0, m = 0, 1, 2, \ldots, \) and

(ii) \( \lim_{m} \Delta_m = 0, \)

is a cyclic vector of \( T_A \).

**Proof.** We first observe that
\[ \inf_{n>0} \|A^nf\| = \mu(f) > 0 \quad \text{for all } f \neq 0. \] (3)
In fact, as \( A \) is power-bounded, let \( \|A^n\| < \delta \), for all \( n = 1, 2, \ldots \). Then \( \mu(f) = 0 \) implies that there exists, for every \( \varepsilon > 0 \), an \( n_0 = n_0(f, \varepsilon) \) such that \( \|A^n f\| < \varepsilon/\delta \); and hence
\[ \|A^n f\| = \|A^n - n_0 A^{n_0} f\| < \|A^{n_0} f\| \|A^{n_0} f\| < \delta \|A^n f\| < \varepsilon \quad \text{for } n \geq n_0. \]

This contradicts our hypothesis that \( A^nf \to 0 \).

Let \( \{e_m\}_{m=0}^\infty \) be the standard orthonormal basis for \( l^2 \) and let \( M = \bigvee_{n=0}^\infty \{T_A^n f\} \). We first show that \( e_0 \in M \). Since
\[ T_A^n f x = \{w_1 w_2 \ldots w_n x_n, w_2 w_3 \ldots w_{n+1} x_{n+1}, \ldots \}, \]
where \( w_n = \|A^n f\| / \|A^n f\|, n = 1, 2, \ldots, \) it follows that
\[
\left\| \frac{T_A^n f x}{w_1 w_2 \ldots w_n x_n} - e_0 \right\|^2 = \sum_{m=n+1}^\infty \left( \frac{w_{m-1} \ldots w_m}{w_1 \ldots w_n} \right)^2 \frac{x_m^2}{x_n^2}
\]
\[
= \sum_{m=0}^\infty \left( \frac{w_{m+2} \ldots w_{m+n+1}}{w_1 \ldots w_n} \right)^2 \frac{x_{m+n+1}}{x_n^2}
\]
\[
\leq \frac{\|A^n f\|^2}{\|A^n f\|^2} \sum_{m>n} \frac{x_m^2}{x_n^2} \Delta_n \quad \text{(by (2))}
\]
\[
\leq \frac{\delta^2 \|f\|^2}{(\mu(f))^2} \Delta_n \quad \text{(by (3))}
\]
\( \to 0, \text{ as } n \to \infty. \)
Thus \( e_0 \in M \). This implies that for each \( n = 1, 2, 3, \ldots \)
\[
y_n = T_{A,f}^n x - w_1 w_2 \cdots w_n x_n e_0
\]
is in \( M \). Proceeding as above, we get
\[
\left\| \frac{y_n}{w_2 w_3 \cdots w_{n+1} x_{n+1}} - e_1 \right\|^2 = \sum_{m=n+2}^{\infty} \left( \frac{w_{m-n+1} \cdots w_m}{w_2 \cdots w_{n+1}} \right)^2 \frac{x_m}{x_{n+1}}^2
\]
\[
= \sum_{m=0}^{\infty} \left( \frac{w_{m+3} \cdots w_{m+n+2}}{w_2 \cdots w_{n+1}} \right)^2 \frac{x_{m+n+2}}{x_{n+1}}^2
\]
\[
= \sum_{m=0}^{\infty} \frac{||A^{m+n+2}f||^2||Af||^2}{||A^{m+2}f||^2||A^{n+1}f||^2} \frac{x_{m+n+2}}{x_{n+1}}^2
\]
\[
< \frac{||A^{n}||^2||Af||^2}{||A^{n+1}f||^2} \sum_{m>n+1} \frac{x_m}{x_{n+1}}^2
\]
\[
< \frac{\delta^2}{(\mu(f))^2} ||Af||^2 \Delta_{n+1}
\]
\[
\to 0, \quad \text{as} \quad n \to \infty.
\]

This gives that \( e_1 \in M \). Now it is obvious by induction that \( e_m \in M \) for all \( m = 0, 1, 2, \ldots, \) and hence \( x \) is cyclic for \( T_{A,f} \).

An operator \( A \) in \( B(H) \) is said to belong to the class \( C_1 \), if it is a contraction (i.e. \( ||A|| < 1 \)) and \( A^nf \to 0 \) for all \( f \neq 0 \). The class \( C_1 \), plays an important role in the study of general contractions [16, p. 72]. We have

**Corollary 2.** Theorem 1 holds for all \( A \in C_1 \).

In our next theorem, we show that the condition \( A^nf \to 0 \) in Theorem 1 can be relaxed in case \( A \) is invertible with a power-bounded inverse.

**Theorem 3.** If \( A \) is invertible and both \( A \) and \( A^{-1} \) are power-bounded, then any vector \( x = (x_m)_{m=0}^{\infty} \) in \( l^2 \) satisfying (*) is a cyclic vector of \( T_{A,f} \).

**Proof.** Since \( A \) and \( A^{-1} \) are both power-bounded, let \( ||A^n|| < \delta \) for all \( n = 0, \pm 1, \pm 2, \ldots \). Now, as in the proof of Theorem 1, it suffices to observe that
\[
\left\| \frac{T_{A,f}^n x}{w_1 w_2 \cdots w_n x_n} - e_0 \right\|^2 < \frac{||A^n||^2||Af||^2}{||A^nf||^2} \sum_{m>n} \frac{x_m}{x_n}^2
\]
\[
= \frac{||A^n||^2||A^{-n}(A^nf)||^2}{||A^nf||^2} \Delta_n
\]
\[
< \frac{||A^n||^2||A^{-n}||^2||A^nf||^2}{||A^nf||^2} \Delta_n
\]
\[
= ||A^n||^2||A^{-n}||^2 \Delta_n
\]
\[
< \delta^4 \Delta_n \to 0, \quad \text{as} \quad n \to \infty.
\]
3. A strictly cyclic operator algebra \( \mathcal{A} \) on \( H \) is a uniformly closed subalgebra of \( B(H) \) such that \( \mathcal{A}f_0 = H \) for some vector \( f_0 \) in \( H \). In this case \( f_0 \) is called a strictly cyclic vector for \( \mathcal{A} \). Moreover, if \( A f_0 = 0 \), \( A \in \mathcal{A} \) implies that \( A = 0 \), we say that \( f_0 \) is a separating vector for \( \mathcal{A} \). The following lemma is due to Embry [5]:

**Lemma E.** Let \( f_0 \) be a strictly cyclic, separating vector for \( \mathcal{A} \). Then there exists a constant \( K \) such that \( \|A\| < K \|Af_0\| \) for every \( A \) in \( \mathcal{A} \).

**Theorem 4.** Let \( \mathcal{A} \) be a strictly cyclic operator algebra with a strictly cyclic separating vector \( f_0 \) and let \( A \in \mathcal{A} \). Then any vector \( x = \{x_m\}_{m=0}^{\infty} \) in \( l^2 \) for which (*)& holds is a cyclic vector for \( TAf_0 \).

For the proof, we see that

\[
\left\| \frac{T^n_{A,f_0}x}{w_1w_2\ldots w_nx_n} - e_0 \right\|^2 < \frac{\|A^n\|^{2}\|f_0\|^2}{\|A^n f_0\|^2} \sum_{m>n} x_m^2 \frac{x_n^2}{x_n^2} \leq K^2 \|f_0\|^2 \Delta_n \quad \text{(by Lemma E)}
\]

\[
\rightarrow 0, \quad \text{as } n \rightarrow \infty.
\]

4. A subalgebra \( \mathcal{A} \) of \( B(H) \) is called transitive if it is weakly closed, contains the identity operator and has no nontrivial invariant subspaces, i.e. there exists no closed subspace \( M \neq \{0\} \) and \( H \) such that \( AM \subset M \) for all \( A \in \mathcal{A} \). \( B(H) \) is obviously a transitive algebra. The question whether there exists a transitive algebra other than \( B(H) \) is an open problem, the so-called transitive algebra problem raised by Kadison [8]. The first solution of the problem was given by Arveson [1]; see also [13]. An elegant account of such solutions is given in Radjavi and Rosenthal [15]. For later additions we refer to Lambert [9] and Yadav and Chatterjee [17]. Here we shall show that if \( A \) is a contraction of the class \( C_{1,\omega} \), then the transitive algebra \( \mathcal{A} \) containing \( T_Af_0 \) is \( B(H) \).

**Theorem 5.** If \( A \) is a contraction belonging to \( C_{1,\omega} \), then the only transitive algebra \( \mathcal{A} \) containing \( T_Af_0 \) is \( B(H) \).

**Proof.** Since \( A \in C_{1,\omega} \), and

\[
\inf_{n>0} \|A^n f\| = \mu(f) > 0 \quad \text{for } f \neq 0,
\]

we have, for all \( n \),

\[
w_1 \ldots w_n \in [\mu(f)/\|f\|, 1].
\]

Thus \( T_Af_0 \) is similar to the multiplicity 1 unilateral shift \( U \). Arveson [1] showed that any transitive algebra containing \( U \) is \( B(H) \). As similarity preserves \( B(H) \) and transitivity, the only transitive algebra \( \mathcal{A} \) containing \( T_Af_0 \) is \( B(H) \).

Finally we state the following theorem without proof:

**Theorem 6.** If \( A \) is an invertible contraction with \( A^{-1} \) power-bounded, then the only transitive algebra \( \mathcal{A} \) containing \( T_Af_0 \) is \( B(H) \).
The authors thank the referee for providing the present short and elegant proof of Theorem 5. The original proof of the theorem was based on the techniques of [13] and ran through several pages.

REFERENCES


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