PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 1, September 1980

κ-FINITENESS AND κ-ADDITIVITY OF MEASURES ON SETS AND LEFT INVARIANT MEASURES ON DISCRETE GROUPS

THOMAS E. ARMSTRONG[1] AND KAREL PRIKRY[2]

ABSTRACT. For any cardinal κ a possibly infinite measure $\mu > 0$ on a set X is strongly non-κ-additive if X is partitioned into κ or fewer μ-negligible sets. The measure μ is purely non-κ-additive if it dominates no nontrivial κ-additive measure. The properties and relationships of these types of measures are examined in relationship to measurable ideal cardinals and real-valued measurable cardinals. Any κ-finite left invariant measure μ on a group G of cardinality larger than κ is strongly non-κ-additive. In particular, σ-finite left invariant measures on infinite groups are strongly finitely additive.

In [2], P. Erdös and R. D. Mauldin showed that the only countably additive, σ-finite, left invariant, positive measure on all the subsets of a group G is the trivial measure. One immediate generalization which suggests itself is that, for an infinite cardinal κ and a group G with $\operatorname{card}(G) > \kappa$, the only κ-additive κ-finite positive left invariant measure is the trivial measure. We establish a stronger statement, even when $\kappa = \omega$, that any κ-finite positive measure on such a group G is strongly non-κ-additive.

DEFINITION 1. Let X be a set and let $\mu \geqslant 0$ be a finitely additive measure on all subsets of X and let κ be a cardinal number.

(i) X, or μ, is *κ-finite* iff it is the union of κ or fewer sets A with $\mu(A) < \infty$.

(ii) X is *κ-negligible* and μ is *strongly non-κ-additive* iff X is the union of κ or fewer sets A with $\mu(A) = 0$.

(iii) μ is *κ-additive* iff whenever $\{A_\alpha: \alpha \in \Gamma\}$ is an increasing family of subsets with $\operatorname{card}(\Gamma) \leqslant \kappa$ then $\mu(\bigcup \{A_\alpha: \alpha \in \Gamma\}) = \sup\{\mu(A_\alpha): \alpha \in \Gamma\}$.

(iv) μ is *purely non-κ-additive* iff the only κ-additive measure it dominates is 0.

DEFINITION 2. A purely non-ω-additive measure is called a *purely finitely additive* measure. A strongly non-ω-additive measure is called a *strongly finitely additive* measure. An ω-finite measure is a *σ-finite measure*.

Only the trivial measure is both κ-additive and purely non-κ-additive. Any strongly non-κ-additive measure is purely non-κ-additive. If μ_1 and μ_2 are finitely additive positive measures $\mu_1 \vee \mu_2$, defined by

$$\mu_1 \vee \mu_2(A) = \sup\{\mu_1(A') + \mu_2(A \setminus A'): A' \subset A\} \quad \text{for } A \subset X,$$

Received by the editors December 12, 1978 and, in revised form, June 13, 1979.

1980 *Mathematics Subject Classification*. Primary 28A12; Secondary 03E55, 28D99.

Key words and phrases. Left invariant measures, left invariant means, κ-additivity, κ-finiteness, pure non-κ-additivity, real-valued measurable cardinal, κ-complete ideal, κ-saturated ideal.

[1]Supported by National Science Foundation Grant MCS 74-05786-A02.

[2]Supported by National Science Foundation Grant MC7 74-06705-A03.

0002-9939/80/0000-0418/$03.00

is the least finitely additive measure dominating μ_1 and μ_2. If μ_1 and μ_2 are κ-additive, κ-finite or strongly non-κ-additive so is $\mu_1 \vee \mu_2$. If $0 \leqslant \nu \leqslant \mu$ and μ is κ-finite or is strongly non-κ-additive so is ν. If $\mu(X) < \infty, 0 \leqslant \nu \leqslant \mu$, and μ is κ-additive so is ν. If $\{\mu_\alpha: \alpha \in \Gamma\}$ is an increasing family of κ-additive measures on X then $\sup\{\mu_\alpha: \alpha \in \Gamma\}$ is also a κ-additive measure on X.

If μ is an arbitrary measure on X the family of κ-additive minorants of μ forms an increasing family whose supremum μ_κ is the largest κ-additive minorant of μ. There is a measure ν on X such that $\mu = \mu_\kappa + \nu$. This measure is defined by

$$\nu(A) = \begin{cases} \mu(A) - \mu_\kappa(A) & \text{if } \mu_\kappa(A) < \infty, \\ \infty & \text{if } \mu_\kappa(A) = \infty. \end{cases}$$

Note that we can denote ν by $\mu - \mu_\kappa$. In general when $\mu_1 \geqslant \mu_2 \geqslant 0$ are measures $\mu_1 - \mu_2$ is defined in the same manner.

In general, if $w \geqslant 0$ is a measure, $\mu = \mu_\kappa + w$ iff $w(A) = (\mu - \mu_\kappa)(A)$ whenever $\mu_\kappa(A) < \infty$. The ensemble of such w is nonempty and decreasing for, if w_1 and w_2 are two such measures, their infimum $w_1 \wedge w_2$, defined by

$$w_1 \wedge w_2(A) = \inf\{w_1(A') + w_2(A \setminus A'): A' \subset A\} \quad \text{for } A \subset X,$$

again satisfies $\mu = \mu_\kappa + w_1 \wedge w_2$. The infimum of the family of w such that $\mu = \mu_\kappa + w$ is the smallest such w. When μ is κ-finite so is μ_κ. In this case, if $0 \leqslant w_0 \leqslant w$ is κ-additive then $\mu_\kappa(A) < \infty$ implies that $w_0(A) = 0$ so w_0 is a κ-additive strongly non-κ-additive measure, hence is 0. Thus, w is purely non-κ-additive.

Definition 3. If $\mu \geqslant 0$ is a measure on X its *κ-additive part* is the largest κ-additive minorant μ_κ. If μ is κ-finite the smallest measure, $\mu_{n\kappa}$, such that $\mu = \mu_\kappa + \mu_{n\kappa}$ is called the *purely non-κ-additive part* of μ.

A κ-finite measure μ is purely non-κ-additive iff $\mu = \mu_{n\kappa}$. In general there are purely non-κ-additive measures which are not strongly non-κ-additive. They are closely tied together in the κ-finite case by this lemma.

Lemma 1. *A κ-finite (σ-finite) measure $\mu \geqslant 0$ on X is purely non-κ-additive iff it is the sum of $\kappa(\omega)$ or fewer strongly non-κ-additive measures.*

Proof. Let $\mu \geqslant 0$ be a finite purely non-κ-additive measure on X. If $\mu \neq 0$ there is a disjoint family $\{E_\lambda: \lambda \in \Lambda\}$ with $\text{card}(\Lambda) \leqslant \kappa$ such that $\Sigma\{\mu(E_\lambda): \lambda \in \Lambda\} < \mu(\bigcup\{E_\lambda: \lambda \in \Lambda\})$. Let $\mu_\lambda(E) = \mu(E \cap E_\lambda)$ for $\lambda \in \Lambda$ and let

$$\mu^1(E) = \mu(E \cap \bigcup\{E_\lambda: \lambda \in \Lambda\}) - \sum\{\mu_\lambda(E): \lambda \in \Lambda\} \quad \text{for } E \subset X.$$

We have $0 \leqslant \mu^1 \leqslant \mu$, $\mu^1(X) > 0$, $\mu^1(E_\lambda) = 0$ for $\lambda \in \Lambda$ and $\mu^1(X \setminus \bigcup\{E_\lambda: \lambda \in \Lambda\}) = 0$. Thus, μ^1 is strongly non-κ-additive. If $\mu_1 = \mu - \mu^1$ then μ_1 is purely non-κ-additive. If $\mu_1(X) > 0$ there is a strongly non-κ-additive $\mu^2 \geqslant 0$ with $\mu^2(X) > 0$ and $\mu^2 \leqslant \mu_1$. We may continue by transfinite induction defining, for ordinals α, $\mu^{\alpha+1}$ as a nonzero strongly non-κ-additive measure less than $\mu - \Sigma_{\beta<\alpha}\mu^\alpha = \mu_\alpha$ where μ^α is defined to be 0 if α is a limit ordinal. This process only terminates if $\mu_{\alpha+1} = 0$ which occurs at some countable ordinal α. Thus, $\mu = \Sigma_{\beta<\alpha}\mu^\alpha$ is the sum of at most ω strongly non-κ-additive measures.

Suppose that μ is κ-finite. There is a partition $\{A_\alpha: \alpha \in \Gamma\}$ of X with $\mu(A_\alpha) < \infty$ if $\alpha \in \Gamma$ with $\operatorname{card}(\Gamma) \leqslant \kappa$. Let $\rho_\alpha(E) = \mu(E \cap A_\alpha)$ for $E \subset X$ and let $\mu'(E) = \Sigma\{\rho_\alpha(E): \alpha \in \Gamma\}$. Since each ρ_α is the sum of countably many strongly non-κ-additive measures μ' is the sum of at most κ strongly non-κ-additive measures. Let $\mu'' = \mu - \mu'$. Since $\mu''(A_\alpha) = 0$ for all α it is strongly non-κ-additive. Since $\mu = \mu' + \mu''$, μ is the sum of at most κ strongly non-κ-additive measures.

Conversely, we wish to show that if μ is the κ-finite sum of κ or fewer strongly non-κ-additive measures $\{\rho_\gamma: \gamma \in \Gamma\}$ then μ is purely non-κ-additive. Let $\{E_\lambda: \lambda \in \Lambda\}$ be a disjoint partition of X with $\operatorname{card}(\Lambda) \leqslant \kappa$ and with $\mu(E_\lambda) < \infty$ if $\lambda \in \Lambda$. Suppose that $0 \leqslant \nu \leqslant \mu$ with ν κ-additive. For $\lambda \in \Lambda$ define $\mu_\lambda(E) = \mu(E \cap E_\lambda)$ and $\nu_\lambda(E) = \nu(E \cap E_\lambda)$ for $E \subset X$. By κ-additivity of ν each ν_λ is κ-additive and $\nu = \Sigma\{\nu_\lambda: \lambda \in \Lambda\}$. If $\nu_\lambda = 0$ for all λ then $\nu = 0$ which will establish the proposition. Thus, by passage to the finite μ_λ it is possible to assume, as we shall, that $\mu(X) < \infty$. Since $\mu(X) < \infty$ there is a countable sequence $\{\rho_i: i \in \omega\} \subset \{\rho_\gamma: \gamma \in \Gamma\}$ such that $\mu = \Sigma_{i\in\omega}\rho_i$. For $n \in \omega$ the partial sum $\rho^n = \Sigma_{i=1}^n \rho_i$ is strongly non-κ-additive and $\rho^n \to \mu$ in total variation norm. Since $\nu = \nu \wedge \mu$ we have $\nu = \lim_{n\to\infty} \nu \wedge \rho^n$ in total variation. Each $\nu \wedge \rho^n$ is strongly non-κ-additive so there is a net $\{A_\pi: \pi \in \Pi\}$ of sets, decreasing to $\varnothing$ with $\operatorname{card}(\Pi) \leqslant \kappa$ and with $(\nu \wedge \rho^n)(A_\pi) = (\nu \wedge \rho^n)(X)$ for $\pi \in \Pi$. Since $\nu(A_\pi) \geqslant (\nu \wedge \rho^n)(A_\pi)$ for all π and since $\nu(A_\pi)\downarrow 0$ along Π, $(\nu \wedge \rho^n)(X) = 0$. Thus, $\nu(X) = \lim_{n\to\infty}(\nu \wedge \rho^n)(X) = 0$ which establishes the lemma for κ-finite measures. □

To establish our main results we shall need a proposition due to Ulam.

DEFINITION 4. An ideal I of subsets of a set X is *κ-additive*, for a cardinal κ, iff whenever a subset F of I contains fewer than κ sets then $\bigcup F \in I$.

One instance of this definition is the ideal, N_μ, of negligible sets for a measure μ. When μ is κ-additive then N_μ is κ^+-additive. (κ^+ is the cardinal succeeding κ.)

DEFINITION 5. Let ρ be a cardinal number and let I be an ideal. I is *ρ-saturated* iff whenever $F \subset \mathcal{P}(X) \setminus I$ consists of disjoint sets then $\operatorname{card}(F) < \rho$.

PROPOSITION 2 (ULAM [7]). *Let X be a set with* $\operatorname{card}(X) = \kappa^+$ *for some κ. No proper κ^+-additive ideal on X which contains singletons is κ^+-saturated.*

Solovay [6] contains more recent results concerning saturated ideals.

PROPOSITION 3. *Let* $\operatorname{card}(X) = \kappa^+$ *for some κ and let $\mu \geqslant 0$ be a κ-finite measure on X such that $\mu(\{x\}) = 0$ for all $x \in X$. μ is then strongly non-κ-additive.*

PROOF. Let I denote the κ^+ additive ideal of subsets of X which are κ-negligible under μ, i.e. all unions of κ or fewer κ-negligible sets. I contains all singletons. Let us show that I is κ^+-saturated. By Proposition 2 this will imply that I is improper hence must contain X, hence will establish the proposition. Let $\mathcal{Q}$ denote a partition of X into at most κ-sets of finite μ-measure. Let $\mathcal{F} \subset \mathcal{P}(X) \setminus I$. Any $Q \in \mathcal{Q}$ meets at most countably many members of $\mathcal{F}$ in a set of strictly positive μ-measure. If $F \in \mathcal{F}$ has $\mu(F \cap Q) = 0$ for $Q \in \mathcal{Q}$ then $F \in I$ which is impossible. Since each F meets some $Q \in \mathcal{Q}$ in a set which is not μ-negligible we may deduce that $\operatorname{card}(\mathcal{F}) \leqslant \omega \cdot \operatorname{card}(\mathcal{Q}) < \kappa^+$. Thus, I is κ^+-saturated. □

COROLLARY 3.1. *For any cardinal κ any κ-finite strongly (purely) non-κ^+-additive measure $\kappa \geqslant 0$ on a set X is strongly (purely) non-κ-additive.*

PROOF. Let $\mu \geqslant 0$ be a κ-finite strongly non-κ^+-additive measure on X. Let $\{X_\lambda: \lambda \in \Lambda\} \subset N_\mu$ partition X with card$(\Lambda) \leqslant \kappa^+$ be such that there is a partition $\mathfrak{Q}$ of Λ with card$(\mathfrak{Q}) \leqslant \kappa$ with $\mu(\bigcup\{X_\lambda: \lambda \in Q\}) < \infty$ for each $Q \in \mathfrak{Q}$. If card$(\Lambda) \leqslant \kappa$, μ is immediately strongly non-κ-additive. Otherwise card$(\Lambda) = \kappa^+$. Define the measure ν on Λ by $\nu(A) = \mu(\bigcup\{X_\lambda: \lambda \in A\})$ for all $A \subset \lambda$. It is immediate that $\nu(\{\lambda\}) = 0$ for $\lambda \in \Lambda$ and, due to the existence of $\mathfrak{Q}$, ν is κ-finite. Proposition 3 implies that ν, hence μ, is strongly non-κ-additive.

Now let $\mu \geqslant 0$ be a κ-finite purely non-κ^+-additive measure on X. Let $\{X_\lambda: \lambda \in \Lambda\}$ be a partition of X with card$(\Lambda) \leqslant \kappa$ and with $\mu(X_\lambda) < \infty$ for $\lambda \in \Lambda$. Set $\mu_\lambda(E) = \mu(E \cap X_\lambda)$ if $\lambda \in \Lambda$ and let $\mu' = \Sigma\{\mu_\lambda: \lambda \in \Lambda\}$. Each μ_λ is purely non-κ^+-additive. By Lemma 1, μ_λ is the sum of at most ω strongly non-κ^+-additive measures each of which, by the preceding paragraph, is strongly non-κ-additive. Since $\mu - \mu'$ is strongly non-κ-additive μ is the sum of at most $\omega \cdot \kappa + 1 = \kappa$ strongly non-κ-additive measures, hence is purely non-κ-additive. □

For any measure μ there is a least cardinal λ for which μ is λ-finite, strongly non-λ-additive, or purely non-λ-additive as long as there is one such λ. There exists a λ for which μ is strongly or purely non-λ-additive iff μ is *diffuse* in that it annihilates all singletons. Corollary 3.1 may be strengthened to say that if μ is strongly (purely) non-κ-additive and is λ-finite for some λ greater than or equal to the largest limit cardinal preceding κ then μ is strongly (purely) non-λ-additive. $\lambda_s(\mu)$ ($\lambda_p(\mu)$) will denote the first cardinal λ for which the diffuse measure μ is strongly (purely) non-λ-additive. Thus, $\lambda_p(\mu) \leqslant \lambda_s(\mu)$ are both limit cardinals. The statements $\lambda_s(\mu) = \aleph_0$ and $\lambda_p(\mu) = \aleph_0$ denote strong and pure finite additivity of μ. $\lambda_s(\mu)$ is the minimum cardinality of a partition of the underlying point set into μ-negligible sets.

The definition of an *Ulam real-valued measurable cardinal* (URVM) is a cardinal $\kappa > \aleph_0$ whose point set admits a diffuse countable additive probability measure. A cardinal $\kappa > \aleph_0$ is *real-valued measurable* (RVM) if it admits a diffuse probability measure which is λ-additive for all $\lambda < \kappa$. Thus, RVM's are URVM's and cardinals larger than the first URVM are URVM [5], [6].

If κ is an RVM and μ is the diffuse probability associated with κ then $\lambda_p(\mu) = \lambda_s(\mu) = \kappa$. If $\{\kappa_n: n \in \omega\}$ are RVM's with associated diffuse probabilities $\{\mu_n: n \in \omega\}$ let $\kappa = \Sigma_{n\in\omega}\kappa_n$ be a disjoint sum and let $\mu = \Sigma_{n\in\omega}2^{-n-1}\mu_n$. The measure μ on κ is a diffuse probability since $\mu \geqslant 2^{-n}\mu_n$ for all n so $\lambda_p(\mu) \geqslant \sup_n\lambda_p(\mu_n) = \kappa$. One also has $\lambda_s(\mu) = \kappa$. This construction shows that, for a diffuse measure μ, $\lambda_p(\mu)$ may be $\aleph_0$ or the sum of countably many RVM's. These are the only possibilities.

PROPOSITION 4. *If μ is a diffuse probability measure either $\lambda_p(\mu)$ is $\aleph_0$ or the sum of countably many RVM's.*

PROOF. We may assume that $\lambda_p(\mu) > \aleph_0$. Set $\lambda_0 = \aleph_0$. If the cardinal $\lambda_\alpha \leqslant \lambda_p(\mu)$ has been defined for the ordinal α let μ_α be the largest λ_α-additive minorant of μ. If

$\mu_\alpha = 0$ let $\lambda_{\alpha+1} = \lambda_\alpha$. We have $\mu_\alpha \neq 0$ iff $\lambda_\alpha < \lambda_p(\mu)$. In this case set $\lambda_{\alpha+1}$ equal to the first cardinal such that μ_α is not $\lambda_{\alpha+1}$-additive. Thus, $\lambda_\alpha < \lambda_{\alpha+1} \leqslant \lambda_p(\mu)$ and $\mu_{\alpha+1} \leqslant \mu_\alpha$ in this case. Furthermore $\mu_\alpha - \mu_{\alpha+1}$ is a nontrivial purely non-$\lambda_{\alpha+1}$-additive measure λ-additive for $\lambda < \lambda_{\alpha+1}$. By Lemma 1, there is a nontrivial strongly non-$\lambda_{\alpha+1}$-additive $\nu \leqslant \mu_\alpha - \mu_{\alpha+1}$ which must be λ-additive for $\lambda < \lambda_{\alpha+1}$. Thus $\lambda_{\alpha+1}$ is an RVM. If α is a limit ordinal set $\lambda_\alpha = \sup_{\beta<\alpha}\lambda_\beta$. The sequence $\{\|\mu_\alpha\|: \alpha$ an ordinal with $\mu_\lambda \neq 0\}$ is strictly decreasing, hence is countable. If α_0 is the first ordinal with $\|\mu_{\alpha_0}\| = \|\mu_{\alpha_0+1}\|$ then $\mu_{\alpha_0} = 0$ and $\lambda_{\alpha_0} = \lambda_p(\mu)$. If α_0 is a successor ordinal $\lambda_p(\mu)$ is a RVM; otherwise $\lambda_p(\mu)$ is a countable sum of RVM's. $\square$

To extend Proposition 4 to κ-finite measures the notion of semifiniteness of measures is needed. A measure μ is said to be *semifinite* if whenever $\mu(A) > 0$ there is an $A' \subset A$ with $0 < \mu(A') < \infty$. The measure μ is said to be *strongly semifinite* iff $\mu(A) = \sup\{\mu(A'): A' \subset A, \mu(A') < \infty\}$. The measure μ is said to be *purely nonsemifinite* iff the only semifinite measure it dominates is 0. For any measure μ define $\mu_1(A)$ to be $\sup\{\mu(A'): A' \subset A, \nu(A') < \infty\}$. Thus, μ_1 is the largest strongly semifinite measure dominated by μ and $\mu_1(A) = \mu(A)$ if $\mu(A) < \infty$. Define $\mu^1(A)$ to be 0 when $\mu(A) < \infty$ and to be ∞ otherwise. We have $\mu_1 + \mu^1 = \mu$. The measure μ^1 is the largest purely nonsemifinite measure dominated by μ. If J is any ideal then $\mu_J(A) = \infty(1 - \chi_J(A))$ is a purely nonsemifinite measure and all such measures arise in this fashion. Any κ-finite κ-additive measure is strongly semifinite. There is a κ-finite purely nonsemifinite measure which is λ-additive for a $\lambda < \kappa$ iff there is a proper λ-additive ideal on κ containing all singletons.

COROLLARY 4.1.

(a) *If μ is a κ-finite diffuse measure either $\aleph_0 \leqslant \lambda_p(\mu) \leqslant \kappa$ or $\lambda_p(\mu)$ is the sum of κ or fewer RVM's.*

(b) *If the measure μ in* (a) *is semifinite $\lambda_p(\mu)$ is either $\aleph_0$ or the sum of κ or fewer RVM's.*

(c) *If μ is a σ-finite diffuse measure then $\lambda_p(\mu)$ is either $\aleph_0$ or the countable sum of RVM's.*

(d) *The first URVM* (*if it exists*) *is the first RVM.*

PROOF. (a) and (b). Partition the underlying point set X into κ or fewer sets $\{A_\alpha: \alpha \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\}$ so that when $\mu_\alpha = \mu|_{A_\alpha}$ then μ_α is finite, so that $\mu_\alpha \neq 0$ for $\alpha \in \Gamma_1 \cup \Gamma_2$, so that $\lambda_p(\mu_\alpha) = \aleph_0$ if $\alpha \in \Gamma_1$, so that $\lambda_p(\mu_\alpha)$ is the countable sum of RVM's for $\alpha \in \Gamma_2$, and so that there is no subset A of $\bigcup\{A_\alpha: \alpha \in \Gamma_3\}$ with $0 < \mu(A) < \infty$. Let $\mu_0 = \Sigma\{\mu_\alpha: \alpha \in \Gamma_1 \cup \Gamma_2\}$ and let $\mu^0 = \mu - \mu_0$ in the usual manner. We have $\lambda_p(\mu^0) \leqslant \operatorname{card}(\Gamma_3) \leqslant \kappa$, $\lambda_p(\mu_0) = \sup\{\lambda_p(\mu_\alpha): \alpha \in \Gamma_1 \cup \Gamma_2\}$ and we have $\lambda_p(\mu) = \lambda_p(\mu^0) \vee \lambda_p(\mu_0)$. When $\Gamma_2 = \varnothing$, $\lambda_p(\mu_0) = \aleph_0$. If $\Gamma_2 \neq 0$, $\lambda_p(\mu_0)$ is the sum of κ or fewer RVM's. This suffices to establish (a). The measure μ is semifinite iff $\Gamma_3 = \varnothing$ in which case (b) follows.

(c) is immediate from (a).

(d) The first URVM, κ_1, is not larger than the first RVM, κ_2. If κ_1 exists let μ be a diffuse countable additive probability on κ_1. We have $\kappa_1 \geqslant \lambda_p(\mu) > \aleph_0$. From (c) it follows that $\lambda_p(\mu) \geqslant \kappa_2$ so $\kappa_1 \geqslant \kappa_2$. $\square$

REMARK. (d) of Corollary 4.1 is well known.

A *measurable ideal cardinal* (MIC) $\kappa > \aleph_0$ is one such that the point set κ admits a diffuse probability μ with N_μ a κ-additive ideal. Any RVM is an MIC. For the measure μ we have $\lambda_s(\mu) = \kappa$ with $\lambda_p(\mu) = \kappa$ iff κ is an MIC. If $\{\kappa_n: n \in \omega\}$ is a countable collection of MIC's with κ as the disjoint sum one can construct a diffuse probability μ on κ with $\lambda_s(\mu) = \kappa$. For finite measures μ, $\lambda_s(\mu)$ can be $\aleph_0$, an MIC or a countable sum of MIC's. These are the only possibilities.

PROPOSITION 5. *If μ is a finite diffuse measure then λ_s must either be $\aleph_0$, an MIC or a countable sum of MIC's.*

PROOF. We may assume that $\lambda_s(\mu) > \aleph_0$. Let λ_1 be the least cardinal so that there is a set $A_1 \notin N_\mu$ admitting a partition by λ_1 sets in $\mathfrak{N}_\mu$. Let $\mu_1 = \mu|A_1$. That $\mathfrak{N}_{\mu_1}$ is λ_1-additive is immediate as is the fact that $\lambda_s(\mu_1) = \lambda_1$. One is able to induce, via μ_1, a diffuse probability v_1 on λ_1 for which N_{v_1} is λ_1-additive. Thus, λ_1 is an MIC. Mimicking the construction of A_1, λ_1, and μ_1 one may construct a finite or countable partition $\{A_n\}$ of the underlying point set X and a corresponding sequence $\{\lambda_n\}$ of MIC's so that $A_n \notin N_\mu$ but A_n is the sum of λ_n sets in N_μ and so that if $\mu_n = \mu|A_n$ then $\lambda_s(\mu_n) = \lambda_n$ for all n. Since $\mu \geqslant \Sigma\mu_n$, $\lambda_s(\mu) \geqslant \Sigma\lambda_s(\mu_n) = \Sigma\lambda_n$. Since A_n is partitioned into λ_n sets in N_μ, $\lambda_s(\mu) \leqslant \Sigma\lambda_n$. Thus, $\lambda_s(\mu) = \Sigma\lambda_n$ which establishes the proposition. $\square$

COROLLARY 5.1. (a) *If μ is a κ-finite diffuse measure then $\lambda_s(\mu)$ is either between $\aleph_0$ and κ or is the sum of κ or fewer MIC's.*

(b) *If μ is a semifinite measure as in* (a) *$\lambda_s(\mu)$ is either $\aleph_0$ or the sum of κ or fewer MIC's.*

(c) *If μ is a σ-finite diffuse measure $\lambda_s(\mu)$ is either $\aleph_0$ or the sum of $\aleph_0$ or fewer MIC's.*

(d) *If X is an infinite set whose cardinal is less than the first MIC any semifinite diffuse measure on X is strongly finitely additive.*

(e) *If the first MIC is less than the first RVM there is a diffuse probability measure with $\aleph_0 = \lambda_p(\mu) < \lambda_s(\mu)$.*

PROOF. (a), (b), (c) have proofs analogous to the proofs of 4.1 (a), (b), and (c).

(d) If κ is the cardinality of X and μ is diffuse then $\lambda_s(\mu) \leqslant \kappa$. (b) and the hypothesis on κ ensure that $\lambda_s(\mu) = \aleph_0$.

(e) Let κ be the first MIC and let μ be the diffuse probability associated with κ as an MIC. We have $\lambda_s(\mu) = \kappa$ but since κ is less than the first RVM 4.1 (c) implies that $\lambda_p(\mu) = \aleph_0$.

We now restrict our attention to measures on a discrete group G. An $\mathfrak{F} \subset \mathfrak{P}(G)$ is said to be *left invariant* iff $F \in \mathfrak{F}$ and $g \in G$ implies that $gF = \{gf: f \in F\} \in \mathfrak{F}$.

PROPOSITION 6. *Let G be a group with* card$(G) > \kappa$. *There are no proper κ^+-additive, κ^+-saturated left invariant ideals in $\mathfrak{P}(G)$.*

PROOF. Select a subgroup G_0 of G with cardinal κ^+. Let the right cosets of G_0 be $\{G_0 h_\alpha : \alpha \in \Gamma\}$ where $H = \{h_\alpha : \alpha \in \Gamma\}$ is a set of coset representatives. The family $G^0 = \{gH : g \in G_0\}$ partitions G into κ^+ sets. If I is a κ^+-saturated ideal then $g_0 H \in I$ for some $g_0 \in G_0$. If I is left invariant as well then $gH \in I$ for all $g \in G_0$. If I is κ^+-additive as well let I_0 denote the ideal of subsets A of G_0 so that $AH \in I$. The ideal I_0 is κ^+-saturated, κ^+-additive and contains singletons; Proposition 2 shows that I_0 is improper, hence that I is improper. □

Proposition 6 is a generalization of the main result of [**2**]. The proof is essentially the same. Proposition 7 provides a closer generalization of their result.

A measure μ on G is *left invariant* iff $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subset G$. When μ is a left invariant probability on G it is called an *invariant mean* [**3**]. Sometimes when invariant means do not exist σ-finite left invariant measures serve as an adequate substitute. If μ is any left invariant measure then N_μ is left invariant as is $\{A : \mu(A) < \infty\}$ and the λ-additive closures of these ideals for any cardinal λ. Thus, a weaker notion than left invariance for a measure μ is that the sets λ-negligible for μ or the sets λ-finite for μ be left invariant. A measure μ for which the λ-negligible sets are left invariant will be said to be λ-weakly left invariant. Intermediate to λ-weakly left invariant and left invariant is the notion of quasi-left-invariance arising in topological dynamics and paraphrased here. The measure μ is *quasi-left-invariant* iff each μ_g for $g \in G$ is mutually absolutely continuous with respect to μ. Here $\mu_g(A) = \mu(gA)$ for all $A \subset G$ and absolute continuity is in the sense of Bochner and Phillips [**1**].

PROPOSITION 7. *Let G be a group with* card(G) *larger than the infinite cardinal* κ. *Any κ-additive κ-weakly left invariant measure μ on G is strongly non-κ-additive.*

PROOF. Since μ is κ-finite and card$(G) > \kappa$, $\mu(\{g\}) = 0$ for some $g \in G$. Thus $\{g\}$ is in the ideal I of κ-negligible sets. By assumption I is left invariant so contains all singletons. As in Propostion 3, I is κ^+-additive and κ^+-saturated. By Proposition 6, I is improper which establishes the proposition. □

COROLLARY 7.1. *An invariant mean μ on an infinite group G is strongly finitely additive.*

PROOF. If card$(G) > \aleph_0$ this is a special case of Proposition 7. If card$(G) = \aleph_0$ an invariant mean μ is diffuse so $\lambda_s(\mu) = \aleph_0$.

PROPOSITION 8. *Let μ be a semifinite, κ-finite κ-weakly left invariant measure on a group G with* card$(G) > \kappa$. *When κ is less than the first RVM (MIC) μ is purely finitely additive (strongly finitely additive).*

PROOF. κ-finiteness of μ and card$(G) > \kappa$ ensures that μ is diffuse. By Proposition 7, $\lambda_p(\mu) \leqslant \lambda_s(\mu) \leqslant \kappa$. That μ is purely (strongly) finitely additive follows from the hypothesis by 4.1(b) (5.1(b)). □

REMARK. We have dealt in this paper only with real or extended real-valued measures. RVM's, URVM's and MIC's arise naturally. Restricting attention to $\{0, 1\}$-valued measures would bring measurable cardinals into play. A cardinal κ is *measurable* if its point set admits a diffuse $\{0, 1\}$-valued, non-κ-additive measure which is λ-additive if λ is a cardinal less than κ. These cardinals are very large, being inaccessible, in fact, even an inaccessible limit of inaccessible cardinals. Any MIC larger than the continuum must be measurable **[5]**. Solovay **[6]** shows that if the existence of measurable cardinals is consistent so is the existence of MIC's and conversely. Solovay **[6]** shows that it is consistent that RVM's exist yet no measurable cardinals exist. Baumgartner, in unpublished notes, shows that if it is consistent that MIC's exist then it is consistent that they exist yet no RVM's exist.

ACKNOWLEDGEMENTS. Suggestions of Fred Galvin and of the referee have greatly contributed to the substance of this paper.

REFERENCES

1. S. Bochner and R. S. Phillips, *Additive set functions and vector lattices*, Ann. of Math. (2) **42** (1941), 316–324.

2. P. Erdös and R. D. Mauldin, *The nonexistence of certain invariant measures*, Proc. Amer. Math. Soc. **59** (1976), 321–322.

3. E. P. Greenleaf, *Invariant means on topological groups*, Van Nostrand, Princeton, N. J., 1969.

4. E. Hewitt and K. Yosida, *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46–66.

5. H. J. Keisler and A. Tarski, *From accessible to inaccessible cardinals*, Fund. Math. **53** (1964), 225–308.

6. R. M. Solovay, *Real-valued measurable cardinals in axiomatic set theory*, Proc. Sympos. Pure Math., vol. 13, Amer. Math. Soc., Providence, R. I., 1971, pp. 397–428.

7. S. M. Ulam, *Zur masstheorie in der allgemeinem Mengenlehre*, Fund. Math. **16** (1930), 144–150.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455