PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 1, September 1980

THE CROSSED PRODUCT OF A C^*-ALGEBRA BY AN ENDOMORPHISM

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ABSTRACT. Let A be a unital, strongly amenable C^*-algebra, $\sigma: A \to pAp$ a $*$-isomorphism (where p is a proper projection of A), and S an isometry such that $SxS^* = \sigma(x)$ for all x in A. If A has no nontrivial σ-invariant ideals, then $C^*(A, S)$ is simple. Furthermore, $C^*(A, S)$ is isomorphic to a corner of the crossed product of $A \otimes$ (compacts) by an automorphism.

J. Cuntz showed in [5] that the C^*-algebra generated by any countable collection of isometries on Hilbert space with range projections summing to 1 is simple. An important step in his proof was the observation that this C^*-algebra is generated by an AF algebra A together with a single isometry S normalizing A (in the sense that SAS^* and S^*AS are both contained in A). Actually, the simplicity of $C^*(A, S)$ does not depend very much on the special circumstances of [5], and in fact follows from fairly mild assumptions on A and S. Our theorem to this effect may be regarded as a generalization of Cuntz's result.

THEOREM 1. *Let A be a strongly amenable unital C^*-algebra acting on a Hilbert space H. Suppose that S is a nonunitary isometry (i.e. $S^*S = 1 \neq SS^*$) in $L(H)$ such that*

(i) *SAS^* and S^*AS are both contained in A; and*

(ii) *the only proper (two-sided) ideal J of A for which $SJS^* \subseteq J$ is the zero ideal.*

Then $C^*(A, S)$ is simple.

Our procedure for proving this is quite similar in outline to that followed in [5]: construct a well-behaved projection of norm one of $C^*(A, S)$ onto A, do the same for the enveloping C^*-algebra B of the $*$-algebra generated by A and S, let the circle group act on B by fixing A and multiplying S by scalars of modulus 1, and then exploit this action to show that the natural map from B to $C^*(A, S)$ is an isomorphism. What is new here is how the desired norm-one projection of $C^*(A, S)$ onto A is shown to exist. This is taken care of in Lemmas 2 and 3 below. After that, the argument proceeds essentially as in [5].

Let A and S be as in Theorem 1. We set $p = SS^*$, so p is a projection in A strictly less than 1. The map $\sigma: A \to A$ defined by $\sigma(x) = SxS^*$ is a $*$-isomorphism of A with pAp, with left inverse σ^* given by $\sigma^*(x) = S^*xS$. For $k \geq 1$, we let

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Received by the editors August 27, 1979.
AMS (MOS) subject classifications (1970). Primary 46L05.
Key words and phrases. C^*-algebra, crossed product, simple.
[1]Research partially supported by NSF grant MCS 77-01850.

$p_k = (S^k)(S^*)^k = \sigma^k(1)$, so $\{p_k\}$ is a decreasing sequence of projections in A. Our notation for the natural left and right actions of a C^*-algebra on its conjugate space is: $(a \cdot g)(b) = g(ba)$, $(g \cdot a)(b) = g(ab)$.

LEMMA 2. *There is a state f_0 on $C^*(A, S)$ whose restriction to A is faithful and which satisfies $f_0(AS^k) = (0)$ for all $k \geqslant 1$.*

PROOF. Our assumption that A is strongly amenable implies that for any state g on a C^*-algebra B containing A, there is a state f in the w^*-closed convex hull of $\{u \cdot g \cdot u^*: u \text{ unitary in } A\}$ which is centralized by A (i.e. $f \cdot x = x \cdot f$ for x in A) [2]. In particular, A has a tracial state. No tracial state of A can vanish at the projection p; the left (= right) kernel of such a state would be a proper ideal containing pAp (= SAS^*), contrary to (ii). This permits us to consider the map of the set of tracial states of A into itself which sends a tracial state τ to the tracial state $\tau(p)^{-1}(\tau \circ \sigma)$. The Schauder fixed point theorem [7] gives us a tracial state τ_0 of A and $0 < r \leqslant 1$ such that $\tau_0 \circ \sigma = r\tau_0$. It follows from (ii) that τ_0 is faithful and hence, because $1 - p \neq 0$, we must have $r < 1$. Extend τ_0 to a state g on $C^*(A, S)$. All of the states $u \cdot g \cdot u^*$ (u unitary in A) also extend τ_0. Using the strong amenability of A, we thus obtain a state extension f of τ_0 to $C^*(A, S)$ which centralizes A. Let K denote the (convex, w^*-compact) set of all such A-centralizing state extensions of τ_0. If f belongs to K, then so does $r^{-1}(S^* \cdot f \cdot S)$, because

$$r^{-1}(S^* \cdot f \cdot S)(x) = r^{-1}\tau_0(\sigma(x)) = \tau_0(x) \qquad (x \text{ in } A)$$

and

$$\begin{aligned}(S^* \cdot f \cdot S)(xY) &= f(SxS^*SYS^*) = f(\sigma(x)SYS^*) \\ &= f(SYS^*\sigma(x)) = (S^* \cdot f \cdot S)(Yx) \qquad (Y \text{ in } C^*(A, S)).\end{aligned}$$

Another application of the Schauder fixed point theorem now yields a state f_0 on $C^*(A, S)$ extending τ_0, centralizing A, and satisfying $f_0 = r^{-1}(S^* \cdot f \cdot S)$. This is the state we want because for x in A and $k \geqslant 1$, we have

$$\begin{aligned}f_0(xS^k) &= r^{-k}f_0\big(S^kxS^k(S^*)^k\big) \\ &= r^{-k}f_0\big(xp_kS^k\big) = r^{-k}f_0(xS^k).\end{aligned}$$

Since $0 < r < 1$, this means that $f_0(xS^k) = 0$, and the lemma is proved.

Let B_0 be the *-algebra generated by A and S. Algebraic manipulations using (i) show that finite sums of the form

$$\sum_1^N (S^*)^k x_{-k} + x_0 + \sum_1^N x_k S^k \qquad (x\text{'s in } A) \tag{*}$$

constitute a *-algebra. (For instance, with $k \geqslant j \geqslant 0$, we have $(S^*)^k x_{-k} x_j S^j = (S^*)^{k-j}[(S^*)^j x_{-k} x_j S^j]$, where the factor in brackets belongs to A.) Hence every operator in B_0 can be written in the form (*).

LEMMA 3. (a) *There is a projection E: $C^*(A, S) \to A$ of norm one satisfying $E(AS^k) = (0) = E((S^*)^k A)$ for $k \geqslant 1$.*

(b) *If X in B_0 is written in the form (*), the coefficients x_j are uniquely determined by X if we require that $x_k p_k = x_k$ and $p_k x_{-k} = x_{-k}$ $(k \geqslant 1)$.*

(c) *If* π: $A \to L(\tilde{H})$ *is a* $*$*-representation and* $\tilde{S}$ *in* $L(\tilde{H})$ *is an isometry such that* $\tilde{S}\pi(x)\tilde{S}^* = \pi(\sigma(x))$, *then* B_0 *and the* $*$*-algebra* $\tilde{B}_0$ *generated by* $\pi(A)$ *and* $\tilde{S}$ *are* $*$*-isomorphic* (*with the isomorphism sending* $x \to \pi(x)$ *and* $S \to \tilde{S}$).

PROOF. (a) It will suffice to show that if X in B_0 is written in the form (*), then $\|X\| \geqslant \|x_0\|$. For this, consider the GNS representation (π_0, ξ_0, H_0) of $C^*(A, S)$ associated with the state f_0 of Lemma 1. Let $H_0' = \overline{\pi_0(A)\xi_0}$. The norm of the compression of $\pi_0(X)$ to H_0' is

$$\sup\{|f_0(w^*Xy)|: w, y \in A, f_0(w^*w) \leqslant 1, f_0(y^*y) \leqslant 1\}.$$

The "zero term" of w^*Xy is w^*x_0y (since $w^*(S^*)^k = (S^*)^k\sigma^k(w^*)$ and $S^ky = \sigma^k(y)S^k$), so $f_0(w^*Xy) = f_0(w^*x_0y)$. The supremum above is therefore just the norm of the image of x_0 under the GNS representation of A arising from $f_{0|_A}$. This is $\|x_0\|$, since $f_{0|_A}$ is faithful, and so we have $\|X\| \geqslant \|\pi_0(X)\| \geqslant \|x_0\|$, proving part (a).

(b) This follows immediately from the observation that $E(X(S^*)^k) = x_kp_k$ and $E(S^kX) = p_kx_{-k}$ $(k \geqslant 1)$.

(c) Notice that the kernel of π is a σ-invariant ideal, so π is faithful by (ii). We have $\tilde{S}\tilde{S}^* = \pi(p)$, and thus $\tilde{S}^*\pi(x)\tilde{S} = \tilde{S}^*\pi(pxp)\tilde{S} = \tilde{S}^*\tilde{S}\pi(\sigma^{-1}(pxp))\tilde{S}^*\tilde{S} = \pi(\sigma^*(x))$ for x in A. Everything we have proved so far about A and S is true also of $\pi(A)$ and $\tilde{S}$. In particular, application of (b) above to B_0 and $\tilde{B}_0$ shows that there is a bijective linear map α: $B_0 \to \tilde{B}_0$ such that $\alpha((S^*)^kx) = (\tilde{S}^*)^k\pi(x)$ and $\alpha(xS^k) = \pi(x)\tilde{S}^k$ for $k \geqslant 0$ and x in A. This is obviously a $*$-map, and a direct computation shows that it is multiplicative.

PROOF OF THEOREM 1. Let B be the completion of B_0 in its greatest C^*-norm. (This makes sense because for any X in B_0 and any $*$-representation π of B_0 on Hilbert space, $\|\pi(X)\|$ does not exceed the sum of the norms of the coefficients x_j in (*).) We have a $*$-monomorphism θ: $A \to B$, an isometry T in B (normalizing $\theta(A)$ in the same way that S normalizes A) such that $B = C^*(\theta(A), T)$, and, because of the universal property of B, a $*$-homomorphism π: $B \to C^*(A, S)$ such that $\pi \circ \theta = \mathrm{id}_A$ and $\pi(T) = S$. Using part (c) of Lemma 3 and, again, the universal property of B, we obtain a homomorphism ρ from the circle group into the group of $*$-automorphisms of B such that $\rho_\lambda(T) = \lambda T$, $\rho_\lambda(\theta(x)) = \theta(x)$ ($|\lambda| = 1$, x in A). Checking first on B_0 shows that the map $\lambda \to \rho_\lambda(Y)$ is norm-continuous for each Y in B. We can therefore define a norm-one projection F: $B \to \theta(A)$ by

$$F(Y) = \int_{|\lambda|=1} \rho_\lambda(Y)\, d\lambda.$$

(The reason that the range of F is precisely $\theta(A)$ is that $F(B_0) = \theta(A)$.) This projection is faithful ($F(Y^*Y) = 0$ implies $Y = 0$) and one checks readily that $\pi \circ F = E \circ \pi$, where E: $C^*(A, S) \to A$ is as in Lemma 3. It is now easy to show that π is an isomorphism. Indeed, we have $\pi(F(\ker \pi)) = E(\pi(\ker \pi)) = (0)$, and since π is injective on $F(B) = \theta(A)$, this means that $F(\ker \pi) = (0)$. But F is faithful, so $\ker \pi = 0$. Suppose finally that π' is an arbitrary $*$-representation of B. By (ii), the restriction of π' to $\theta(A)$ is faithful, so $\pi'(B)$ is generated by an isomorphic copy of A and a normalizing isometry $\pi(S)$. Using the norm-one projection E': $\pi'(B) \to \pi'(\theta(A))$ that one obtains from Lemma 3 in place of E in the argument above, we deduce that π' must be faithful. Hence $C^*(A, S)$, which is isomorphic to B, is simple.

The algebra $C^*(A, S)$ is always amenable [9], hence nuclear [4] (see also [3]), but does not admit a tracial state and therefore cannot be strongly amenable. One can think of $C^*(A, S)$ as the crossed product of A by the endomorphism σ. Indeed, given any "σ-covariant" pair $(\pi, \tilde{S})$ as in Lemma 3(c), that lemma and the proof of Theorem 1 ensure that $C^*(A, S)$ and $C^*(\pi(A), \tilde{S})$ are isomorphic in a natural way. We remark that if A is any unital C^*-algebra, p a nonzero projection in A, and σ: $A \to pAP$ a $*$-isomorphism (should one exist), then there exists a σ-covariant representation of A. An easy way to see this is to define σ^*: $A \to A$ by $\sigma^*(x) = \sigma^{-1}(pxp)$, note that composition with σ^* takes states of A to states, and use the Schauder fixed point theorem to obtain a state f of A such that $f \circ \sigma^* = f$ (and hence $f \circ \sigma = f$). If (π, ξ, H) is the GNS representation of A associated with f, one checks readily that the equation $\tilde{S}\pi(x) = \sigma(x)\xi$ defines an isometry $\tilde{S}$ satisfying $\tilde{S}\pi(x)\tilde{S}^* = \pi(\sigma(x))$ for x in A.

It turns out that under the hypotheses of Theorem 1, $C^*(A, S)$ can be obtained from a crossed product of the standard sort. An explicit construction is given in [5] to show this for the situation considered there; our result below is of necessity somewhat less detailed. Here, $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space.

THEOREM 2. *Let A and S be as in Theorem 1. There exist a $*$-automorphism θ of $A \otimes \mathcal{K}$ and a projection Q in the crossed product $C^*(A \otimes \mathcal{K}, \theta)$ such that $C^*(A, S)$ is isomorphic to $QC^*(A \otimes \mathcal{K}, \theta)Q$.*

PROOF. Let $B = C^*(A, S)$. From the proof of Theorem 1, we know that there is a continuous action ρ of the circle group on B such that $\rho_\lambda(x) = x$ and $\rho_\lambda(S) = \lambda S$ ($|\lambda| = 1$, x in A). Recall that the crossed product $C^*(B, \rho)$ of B by the action ρ is the completion in the greatest C^*-norm of $C(T, B)$, the space of continuous B-valued functions on the circle group, considered as a $*$-algebra with multiplication and involution defined by

$$(FG)(\lambda) = \int_{|\mu|=1} F(\mu)\rho_\mu(G(\bar{\mu}\lambda))\, d\mu \qquad (F^*)(\lambda) = \rho_\lambda\big(F(\bar{\lambda})^*\big).$$

Let P in $C(T, B)$ be the function with constant value 1. It is immediate that P is a projection, and, using the fact that A is the fixed-point algebra of (B, ρ), one checks (see [10]) that A and $PC^*(B, \rho)P$ are isomorphic via the map that sends x in A to the function in $C(T, B)$ with constant value x. We now claim that the closed two-sided ideal L of $C^*(B, \rho)$ generated by P is all of $C^*(B, \rho)$. For this, consider products of the form FPG, where F is the function constantly equal to some given X in B, and $G(\lambda) = \rho_\lambda(Y)$ for some Y in B. The convolution formula gives $(FPG)(\lambda) = X\rho_\lambda(Y)$. In particular, if $X = (S^*)^k$ and $Y = yS^j$ ($k, j \geq 0, y$ in A), then if $j \geq k$, we have $(FPG)(\lambda) = \lambda^j(\sigma^*)^k(y)S^{j-k}$, while if $j < k$, we obtain $(FPG)(\lambda) = \lambda^j(S^*)^{k-j}(\sigma^*)^j(y)$. Since σ^* is surjective, it follows that all functions of the form

$$\lambda \to \lambda^m (S^*)^n x \quad \text{or} \quad \lambda \to \lambda^m x S^n \qquad (n \geqslant 0, m \geqslant 0, x \text{ in } A)$$

belong to L. To show that $m < 0$ is also allowed here, take $X = xS^k$ and $Y = (S^*)^j y$ and suppose that $j \geqslant k$. We have

$$(FPG)(\lambda) = X\rho_\lambda(Y) = \lambda^{-j} x p_k (S^*)^{j-k} y$$
$$= \lambda^{-j}(S^*)^{j-k}\sigma^{j-k}(x)p_j y.$$

Now as x ranges over A, $(S^*)^{j-k}\sigma^{j-k}(x)p_j$ ranges over $(S^*)^{j-k}p_{j-k}Ap_{j-k}p_j$ (= $(S^*)^{j-k}Ap_j$). By part (ii) of the assumption on A and S, the set $\{wp_j y\colon w, y \in A\}$ spans A. (This is because $\sigma(p_j) = p_{j+1} \leqslant p_j$.) For $n \geqslant 0$, $m < 0$, and any x in A, the function $\lambda \to \lambda^m(S^*)^n x$ belongs to L. A similar argument with $j < k$ shows that the same is true of the function $\lambda \to \lambda^m x S^n$. It now follows that for any continuous $f\colon T \to \mathbf{C}$, the functions

$$\lambda \to f(\lambda)(S^*)^n x \quad \text{and} \quad \lambda \to f(\lambda) x S^n$$

belong to L. A straightforward partition-of-unity argument shows that these functions span a sup-norm dense subspace of $C(T, B)$. But the sup-norm on $C(T, B)$ dominates the L^1-norm, which in turn dominates the greatest C^*-norm, so $L = C^*(B, \rho)$ as claimed. In other words, the "corner" $PC^*(B, \rho)P$ of $C^*(B, \rho)$ to which A is isomorphic is a full corner. By Corollary 2.6 of **[1]**, then, $A \otimes \mathcal{K}$ and $C^*(B, \rho) \otimes \mathcal{K}$ are isomorphic. (Application of this result requires that $C^*(B, \rho)$ have a strictly positive element. This is not a problem here. Any scalar-valued function in $C(T, B)$ all of whose Fourier coefficients are positive is a strictly positive element of $C^*(B, \rho)$.) Let $\hat{\rho}$ be the $*$-automorphism that generates the action dual to ρ on $C^*(B, \rho)$ (see **[11]**), and let θ be the automorphism of $A \otimes \mathcal{K}$ corresponding to $\hat{\rho} \otimes \mathrm{id}_{\mathcal{K}}$. By H. Takai's duality theorem **[11]**, the crossed product $C^*(A \otimes \mathcal{K}, \theta)$ is isomorphic to $B \otimes \mathcal{K} \otimes \mathcal{K}$, so B is isomorphic to $QC^*(A \otimes \mathcal{K}, \theta)Q$ for an appropriate projection Q in $C^*(A \otimes \mathcal{K}, \theta)$.

We conclude with a remark about KMS states for what may be termed the "natural" dynamics on $C^*(A, S)$. Let A, S, σ, E, and ρ be as in Theorem 1 and its proof. (Regard ρ as a 2π-periodic action of $\mathbf{R}$ on $C^*(A, S)$: $\rho_t(x) = x$ for x in A, $\rho_t(S) = \exp(it)S$.) An appropriate modification of the argument in **[8]** shows that for $0 < \beta < \infty$, the β-KMS states (if there are any) of the C^*-dynamical system $(C^*(A, S), \rho)$ are precisely those of the form $\tau \circ E$, where τ is a tracial state on A such that $\tau \circ \sigma = \exp(-\beta)\tau$. (See also the last paragraph of **[6]**.) In light of this, it would appear that the algebras we have investigated here could serve as a useful source of examples of C^*-dynamical systems exhibiting various sorts of unusual KMS-phenomena.

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