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SIMPLE PROOF OF A THEOREM OF ERDÖS AND LAX

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ABSTRACT. An elementary new and simple proof of Erdös-Lax theorem is given which in essence involves no analysis.

Let $P(z)$ be a polynomial of degree n with $\operatorname{Max}_{|z|=1}|P(z)| = 1$, then

$$\operatorname*{Max}_{|z|=1}|P'(z)| \leqslant n. \tag{1}$$

Inequality (1) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for reference see [5]). Inequality (1) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, P. Erdös conjectured and later P. D. Lax [3] proved the following

THEOREM A. *If $P(z)$ is a polynomial of degree n with* $\operatorname{Max}_{|z|=1}|P(z)| = 1$ *and $P(z)$ has no zeros in the disk $|z| < 1$, then*

$$\operatorname*{Max}_{|z|=1}|P'(z)| \leqslant n/2. \tag{2}$$

The result is best possible and equality in (2) holds for $P(z) = (\alpha + \beta z^n)/2$, where $|\alpha| = |\beta| = 1$.

For other proofs of Theorem A see [1], [2], and [4]. In this paper we give an apparently new proof of Theorem A, which in essence involves no analysis. The proof depends on the following lemma which is also of independent interest.

LEMMA 1. *If $P(z)$ is a polynomial of degree n and $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + a$, where $a \neq -1$ is any nonzero complex number, then for any complex number t,*

$$tP'(t) = \frac{n}{1+a}P(t) + \frac{1+a}{na}\sum_{k=1}^{n} P(tz_k)\frac{z_k}{(z_k-1)^2}. \tag{3}$$

PROOF OF LEMMA 1. Let t be an arbitrary complex number. Consider the function $F_t(z)$ defined by $F_t(z) = (P(tz) - P(t))/(z-1)$. Then $F_t(z)$ is a polynomial of degree $\leqslant n-1$, and therefore, by using Lagrange's interpolation formula with $z_1, z_2, \ldots, z_n$ as the basic points of interpolation we can write $F_t(z)$ as

$$F_t(z) = \sum_{k=1}^{n} F_t(z_k)\frac{z^n + a}{nz_k^{n-1}(z - z_k)}.$$

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Since $z_k^n + a = 0$ and $z_k \neq 0$, $k = 1, 2, \ldots, n$, therefore, $z_k^{n-1} = -a/z_k$ and we get

$$F_t(z) = (1/na) \sum_{k=1}^{n} F_t(z_k) \frac{z_k(z^n + a)}{(z_k - z)}.$$

Now using the fact $F_t(1) = tP'(t)$, we obtain the following identity in t.

$$tP'(t) = \frac{1}{na} \sum_{k=1}^{n} F_t(z_k) \frac{z_k(1+a)}{(z_k - 1)} = \frac{1+a}{na} \sum_{k=1}^{n} (P(tz_k) - P(t)) \frac{z_k}{(z_k - 1)^2}$$

$$= \frac{1+a}{na} \sum_{k=1}^{n} P(tz_k) \frac{z_k}{(z_k - 1)^2} - \frac{(1+a)P(t)}{na} \sum_{k=1}^{n} \frac{z_k}{(z_k - 1)^2}. \tag{4}$$

Setting $P(t) = t^n$ in (4) we get

$$(1/n) \sum_{k=1}^{n} z_k/(z_k - 1)^2 = \frac{-na}{(1+a)^2}. \tag{5}$$

Combining (4) and (5) we get (3) and therefore the lemma is established.

From Lemma 1, we now deduce the following

LEMMA 2. *If $P(z)$ is a polynomial of degree n, then*

$$|P'(z)| + |nP(z) - zP'(z)| \leqslant n \operatorname*{Max}_{|z|=1} |P(z)| \quad \textit{for } |z| = 1. \tag{6}$$

PROOF OF LEMMA 2. In Lemma 1 we take $a \neq -1$ to be an arbitrary complex number such that $|a| = 1$, then the zeros z_k of $z^n + a$ are of unit modulus and $z_k \neq 1$, $k = 1, 2, \ldots, n$. So that from (3) for $|t| = 1$ we obtain

$$|atP'(t) + tP'(t) - nP(t)| = \left| \frac{(1+a)^2}{na} \sum_{k=1}^{n} P(tz_k) z_k/(z_k - 1)^2 \right|$$

$$\leqslant \left| \frac{(1+a)^2}{na} \right| \sum_{k=1}^{n} |z_k/(z_k - 1)^2| \operatorname*{Max}_{|t|=1} |P(t)|. \tag{7}$$

Now if $|z| = 1$ and $z \neq 1$ then $z/(z-1)^2$ is a negative real number. In fact it can be easily seen that $e^{i\theta}/(e^{i\theta} - 1)^2 = -1/4 \sin^2\theta/2$, $\theta \not\equiv O(2\pi)$ and moreover for $|a| = 1$ and $a \neq -1$, $(1+a)^2/a$ is a positive real number. Therefore,

$$\left| \frac{(1+a)^2}{na} \right| \sum_{k=1}^{n} |z_k/(z_k - 1)^2| = -\frac{(1+a)^2}{na} \sum_{k=1}^{n} z_k/(z_k - 1)^2$$

$$= n \quad \text{(by (5))}.$$

Hence from (7) it follows that

$$|atP'(t) + tP'(t) - nP(t)| \leqslant n \operatorname*{Max}_{|t|=1} |P(t)| \quad \text{for } |t| = 1, |a| = 1, a \neq -1.$$

This equality obviously holds for $a = -1$ also. Choosing the argument of a such that

$$|atP'(t) + tP'(t) - nP(t)| = |P'(t)| + |nP(t) - tP'(t)|,$$

we get

$$|P'(z)| + |nP(z) - zP'(z)| \leqslant n \operatorname*{Max}_{|z|=1} |P(z)| \quad \text{for } |z| = 1.$$

This is equivalent to the desired result.

PROOF OF ERDÖS-LAX THEOREM. Since the polynomial $P(z)$ does not vanish in the disk $|z| < 1$, we can write $P(z) = c\prod_{j=1}^{n}(z - w_j)$ where $|w_j| \geqslant 1, j = 1, 2, \ldots, n$. Now for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$ other than the zeros of $P(z)$ we have

$$\begin{aligned}\operatorname{Re}(e^{i\theta}P'(e^{i\theta}))/nP(e^{i\theta}) &= \frac{1}{n}\sum_{j=1}^{n} \operatorname{Re}\left(e^{i\theta}/\left(e^{i\theta} - w_j\right)\right) \\ &\leqslant (1/n)\sum_{1}^{n}(1/2) = 1/2.\end{aligned}$$

This implies

$$|e^{i\theta}P'(e^{i\theta})/nP(e^{i\theta})| \leqslant |1 - (e^{i\theta}P'(e^{i\theta}))/nP(e^{i\theta})|$$

for points $e^{i\theta}$ other than the zeros of $P(z)$. Equivalently

$$|P'(e^{i\theta})| \leqslant |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| \tag{8}$$

for points $e^{i\theta}$ other than the zeros of $P(z)$. Since the inequality (8) is trivially true for points $e^{i\theta}$ which are the zeros of $P(z)$, therefore, it follows that

$$|P'(z)| \leqslant |nP(z) - zP'(z)| \quad \text{for } |z| = 1. \tag{9}$$

Combining the inequality (9) with the conclusion of Lemma 2, we get

$$|P'(z)| \leqslant (n/2) \operatorname*{Max}_{|z|=1} |P(z)| \quad \text{for } |z| = 1.$$

This is equivalent to the desired result.

If $P(z)$ is a polynomial of degree n, then obviously

$$|nP(z) - zP'(z)| + |zP'(z)| \geqslant n|P(z)|.$$

This gives

$$\operatorname*{Max}_{|z|=1}(|nP(z) - zP'(z)| + |P'(z)|) \geqslant n \operatorname*{Max}_{|z|=1} |P(z)|.$$

With the help of this inequality we can restate Lemma 2 as follows.

LEMMA 3. *If $P(z)$ is a polynomial of degree n, then*

$$\operatorname*{Max}_{|z|=1}(|P'(z)| + |nP(z) - zP'(z)|) = n \operatorname*{Max}_{|z|=1} |P(z)|. \tag{10}$$

REMARK 1. If $P(z)$ is a self-inversive polynomial, i.e. if $P(z) = Q(z)$, where $Q(z) = z^n\overline{P(1/\bar{z})}$, then $|Q'(z)| = |nP(z) - zP'(z)|$ for $|z| = 1$ and it easily follows from Lemma 3 that

$$\operatorname*{Max}_{|z|=1} |P'(z)| = (n/2) \operatorname*{Max}_{|z|=1} |P(z)|.$$

Many other interesting results follow easily from Lemma 1. For example the following theorem is an immediate consequence of Lemma 1.

THEOREM. *If $P(z)$ is a polynomial of degree n such that $P(\beta) = 0$, then*

$$|\beta P'(\beta)| \leqslant (n/2) \operatorname*{Max}_{1 \leqslant k \leqslant n} |P(\beta z_k)|$$

where z_k, $k = 1, 2, \ldots, n$, are the zeros of $z^n + 1$. The result is sharp.

COROLLARY. *If $P(z)$ is a polynomial of degree n such that $P(1) = 0$, then*

$$|P'(1)| \leqslant (n/2) \operatorname*{Max}_{1 \leqslant k \leqslant n} |P(z_k)|$$

where z_k, $k = 1, 2, \ldots, n$, are the zeros of $z^n + 1$. The result is best possible and equality holds for $P(z) = z^n - 1$.

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