£ *(K) AND OTHER LATTICES OF RECURSIVELY ENUMERABLE SETS

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Abstract. We study the direct product operation on lattices which are principal filters of $\mathcal{S}^*$, the lattice of r.e. sets modulo finite sets, to generate new isomorphism types of such filters and to characterize the one generated by the complete r.e. set $K$.

A major trend in the long term project of analyzing the lattice $\mathcal{S}$ of recursively enumerable sets and $\mathcal{S}^*$ its quotient modulo the finite sets has been the investigation of the class $\mathcal{F}$ of principal filters of $\mathcal{S}^*$, i.e. of the lattices $\mathcal{L}^*(A) = \{B \in \mathcal{S}^* | B \subseteq A\}$ for r.e. $A$. (Note that $A \subseteq B$ if $A \triangle B$ is finite.) Of course the principal ideals of $\mathcal{S}^*$ are irrelevant since $\{B \in \mathcal{S}^* | B \subseteq A\} \approx \mathcal{S}$ for every $A \neq \emptyset$. The first such conscious investigations began with Myhill [1956] who defined maximal r.e. sets, i.e. sets $M$ such that $\mathcal{L}^*(M) \approx \{0, 1\}$ (the two element Boolean algebra). Indeed the hyperhypersimple sets of Post [1944], although defined in terms of the intersection of arrays with the sets complement, also turned out to be related to this line of thought. Lachlan [1968] showed that they are precisely the r.e. sets $A$ such that $\mathcal{L}^*(A)$ is a Boolean algebra. He was also able to completely characterize the members of $\mathcal{F}$ which are Boolean algebras as exactly the $\Sigma_3$ presentable ones.

At the other extreme one finds the $r$-maximal sets. These are easily seen to be equivalent to those with $\mathcal{L}^*(A)$ having no complemented elements. Classifying the isomorphism types of the $r$-maximal sets however seems to be a difficult open problem. The only other commonly recognized principal filter in $\mathcal{S}^*$ is the nearly ubiquitous one $\mathcal{S}^*$ itself. Of course if $A$ is recursive it is immediate that $\mathcal{L}^*(A) \approx \mathcal{S}^*$ but Soare [1974], [1981] has shown that this type is extremely common: If $A$ is an r.e. infinite set and $A$ is semilow (i.e. $\{e | W_e \cap \overline{A} \neq \emptyset\} < \emptyset\}$ then $\mathcal{L}^*(A) \approx \mathcal{S}^*$. This means that there are r.e. sets $A$ in every r.e. degree with $\mathcal{L}^*(A) \approx \mathcal{S}^*$ and all low r.e. sets $A$ (i.e., $A' < \mathcal{T} \emptyset$) have this property.

Our goal here is simply to provide some additional examples of types of principal filters in $\mathcal{S}^*$. We will do this by describing some simple properties of the direct product of lattices in $\mathcal{F}$. We will then use it to generate new isomorphism types in $\mathcal{F}$. In addition these properties will enable us to make one really new identification. We will characterize the isomorphism type of $\mathcal{L}^*(K)$ by an absorption property.

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with respect to products in $\mathcal{L}$. Some connections between the structure of $\mathcal{L} \ast(A)$ and the degree of $A$ will also be pointed out.

Our starting point is a simple fact from lattice theory. We work with distributive lattices with 0 and 1. Basic references are Birkhoff [1948] for lattice theory and Rogers [1967] for recursion theory. An excellent current survey of r.e. sets and degrees is Soare [1978].

**Lemma 1.** $L_1 \otimes L_2 \cong L$ iff there are $x_1$ and $x_2$ in $L$ such that $x_1 \land x_2 = 0$, $x_1 \lor x_2 = 1$ and $L_i \cong L(x_i)$ where $L(x_i) = \{ y \in L \mid y > x_i \}$.

**Proof.** The idea is just that $x_1, x_2$ are the images of $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$ respectively. See Birkhoff [1948, p. 26].

Our first observation is that $S$ is closed under products. Consider $t \ast(Ax)$ and $\ell \ast(A2)$. Let $R$ be an infinite coinfinite recursive set with complement $\bar{R}$. We map $f_1: \mathbb{N} \to R, f_2: \mathbb{N} \to \bar{R}$ by one-one onto recursive maps. It is then immediate that $\ell \ast(R \cup f_1[A_1]) \cong \ell \ast(A_1)$ and $\ell \ast(R \cup f_2[A_2]) \cong \ell \ast(A_2)$. Thus

$$\ell \ast(A_1) \otimes \ell \ast(A_2) \cong \ell \ast(R \cup f_1[A_1]) \otimes \ell \ast(R \cup f_2[A_2])$$

but by the lemma this is just $\ell \ast(f_1[A_1] \cup f_2[A_2])$. Note that if $A_1$ and $A_2$ are simple so is $f_1[A_1] \cup f_2[A_2]$. Thus the class of principal filters generated by simple sets is also closed under direct product. Of course $\ell \ast(\mathbb{N}) = 1$, the trivial one-element lattice, is an identity for products in $\mathcal{F}$.

We next consider $S \ast$ and see that it is an indecomposable idempotent.

**Corollary 2.** $S \ast \otimes S \ast \cong S \ast$.

**Proof.** Let $x_1$ and $x_2$ be given by any infinite coinfinite recursive set and its complement.

**Corollary 3.** If $L_1 \otimes L_2 \cong S \ast$ then $L_i = 1$ or $S \ast$.

**Proof.** By the lemma the $L_i$ are isomorphic to $\ell \ast(A_i)$ for $A_1 \cap A_2 = \ast \emptyset$ and $A_1 \cup A_2 = \ast \mathbb{N}$. Thus the $A_i$ are recursive and $\ell \ast(A_i) \cong S \ast$ or 1 (if one is N).

Thus products of $S \ast$ give no new isomorphism types in $\mathcal{F}$ and of course products of Boolean algebras are still Boolean algebras. We can however combine these two known types to generate new ones. We use a more general version of Lemma 1 to prove that one gets new types in this way.

**Theorem 4.** If $S \ast \otimes L_1 \cong S \ast \otimes L_2$ then $L_1 \cong L_2$ or $L_1 \cong S \ast \otimes L_2$ or $L_2 \cong S \ast \otimes L_1$.

**Proof.** By Theorem 7 on p. 26 of Birkhoff [1948] there are lattices $Z_1, Z_2$, $Z_1^2, Z_2^2$ such that $Z_1 \otimes Z_1^2 \cong S \ast$, $Z_1 \otimes Z_2 \cong S \ast$, $Z_2 \otimes Z_2^2 \cong L_1$ and $Z_1 \otimes Z_2^2 \cong L_2$. By Corollary 3, $Z_1, Z_2$ and $Z_1^2, Z_2^2$ are 1 or $S \ast$. If $Z_1 \cong 1$ then $Z_2 \cong S \ast \otimes Z_2^2$ and so $L_1 \cong S \ast \otimes Z_2^2 \cong L_2$ as required. Suppose now that $Z_1 \cong S \ast$. If $Z_2 \cong Z_2^2$ ($\cong 1$ or $S \ast$) then again $L_1 \cong L_2$ ($\cong Z_2^2$ or $S \ast \otimes Z_2^2$ respectively). The remaining cases are $(Z_2 \cong 1 \& Z_2^2 \cong S \ast)$ and $(Z_2 \cong S \ast \otimes Z_1 \cong 1)$. In the first case $L_1 \cong Z_2^2$ and $L_2 \cong S \ast \otimes Z_2^2$ as required. The second of course gives $L_2 \cong Z_2^2$ and $L_1 \cong S \ast \otimes Z_2^2$. □
Corollary 5. If $\mathfrak{B}_1 \cong \mathfrak{B}_2$ are Boolean algebras in $\mathfrak{T}$ then $\mathbb{E}^* \otimes \mathfrak{B}_1$ and $\mathbb{E}^* \otimes \mathfrak{B}_2$ are nonisomorphic elements of $\mathfrak{T}$. Neither is a Boolean algebra or $\mathbb{E}^*$ but both are isomorphic to principal filters generated by simple sets.

Proof. As $\mathbb{E}^*$ is not a Boolean algebra $\mathfrak{B}_i \cong \mathbb{E}^* \otimes \mathfrak{B}_i$ and we apply the theorem. As there is a simple set $A$ with $\mathbb{E}^*(A) \equiv \mathbb{E}^*$ the $\mathbb{E}^* \otimes \mathfrak{B}_i$ are isomorphic to principal filters generated by simple sets by the closure of this class under direct product.

We next want to point out a simple relation between products of elements of $\mathfrak{F}$ and the degrees of the r.e. sets to which they correspond.

Lemma 6. If $\mathbb{E}^*(A) \equiv \mathbb{E}^*(A_1) \otimes \mathbb{E}^*(A_2)$ then there are $B_i$ with $\mathbb{E}^*(B_i) \equiv \mathbb{E}^*(A_i)$ and $B_1 \oplus B_2 \equiv_T A$.

Proof. The elements of $\mathbb{E}^*(A)$ guaranteed by our basic lemma are now r.e. sets $B_1$ and $B_2$ with $B_1 \cap B_2 = A$, $B_1 \cup B_2 = N$ and $\mathbb{E}^*(B_i) \equiv \mathbb{E}^*(A_i)$. To see if $x \in B_i$ ask if $x \in A$. If so, $x \in B_i$. If not, enumerate both $B_1$ and $B_2$ until $x$ appears in one of them. If it first appears in $B_1$, $x \in B_1$ and otherwise $x \not\in B_i$. Of course $x \in A$ iff $x \in B_1$ and $x \in B_2$.

Thus restrictions on $\mathbb{E}^*(A)$ that push the degree of $A$ upward are passed on by products. So we have for example

Corollary 7. If $\mathfrak{B} \not\equiv 1$ is a Boolean algebra and $\mathbb{E}^*(A) \equiv \mathbb{E}^* \otimes \mathfrak{B}$ then $A$ is high i.e. $\mathfrak{D}^\prime \equiv_T A'$.

Proof. By Lachlan [1968] any $B$ with $\mathbb{E}^*(B) \equiv \mathfrak{B}$ is hyperhypersimple. Martin [1966] then shows that $B$ must be high.

Carrying this idea to an extreme one might guess that the most complicated sets $A$ should have $\mathbb{E}^*(A)$'s with the most factors. Indeed this gives us our characterization of $\mathbb{E}^*(K)$. ($K = \{e | e \in W_e\}$ is of course a 1-complete r.e. set and so in many ways the most complicated one.)

Theorem 8. $\mathbb{E}^*(K) \equiv \mathbb{E}^*(K) \otimes \mathbb{E}^*(A)$ for every r.e. $A$.

Proof. Let $R_1, f_1$ and $f_2$ be as in the proof that $\mathfrak{T}$ is closed under products following Lemma 1. As before we see that $\mathbb{E}^*(K) \otimes \mathbb{E}^*(A) \equiv \mathbb{E}^*(f_1[K] \cup f_2[A])$. As $f_1$ is a recursive one-one and onto map of $N$ to a recursive set $R,f_1[K]$ is also a complete set as is $f_1[K] \cup f_2[A]$. Thus by Myhill [1955] $K$ and $f_1[K] \cup f_2[A]$ are recursively isomorphic and so

$$\mathbb{E}^*(K) \equiv \mathbb{E}^*(f_1[K] \cup f_2[A]) \equiv \mathbb{E}^*(K) \otimes \mathbb{E}^*(A)$$

as required. □

This theorem characterizes the isomorphism type of $\mathbb{E}^*(K)$ for if $\mathbb{E}^*(B) \equiv \mathbb{E}^*(B) \otimes \mathbb{E}^*(A)$ for every $A$ then $\mathbb{E}^*(B) \equiv \mathbb{E}^*(B) \otimes \mathbb{E}^*(K) \equiv \mathbb{E}^*(K)$. Moreover our earlier results show that the type of $\mathbb{E}^*(K)$ is not any of the ones considered before, i.e., it is not generated as a product of lattices which are Boolean algebras or $\mathbb{E}^*$. Finally our results on degrees show that if $\mathbb{E}^*(K) \equiv \mathbb{E}^*(A)$ then $A$ is high.
BIBLIOGRAPHY


——— [1956], *The lattice of recursively enumerable sets*, J. Symbolic Logic 21, 220.


——— [1981], *Automorphisms of the lattice of recursively enumerable sets*, Part II: Low sets (to appear).


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