**£*(K) AND OTHER LATTICES OF RECURSIVELY ENUMERABLE SETS**

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**Abstract.** We study the direct product operation on lattices which are principal filters of \( \mathcal{S}^* \), the lattice of r.e. sets modulo finite sets, to generate new isomorphism types of such filters and to characterize the one generated by the complete r.e. set \( K \).

A major trend in the long term project of analyzing the lattice \( \mathcal{S} \) of recursively enumerable sets and \( \mathcal{S}^* \) its quotient modulo the finite sets has been the investigation of the class \( \mathcal{T} \) of principal filters of \( \mathcal{S}^* \), i.e. of the lattices \( \mathcal{L}^*(A) = \{ B \in \mathcal{S}^* | B \supseteq A \} \) for r.e. \( A \). (Note that \( A \subseteq^* B \) iff \( A \triangle B \) is finite.) Of course the principal ideals of \( \mathcal{S}^* \) are irrelevant since \( \{ B \in \mathcal{S}^* | B \subseteq^* A \} = \mathcal{S}^* \) for every \( A \neq^* \emptyset \). The first such conscious investigations began with Myhill [1956] who defined maximal r.e. sets, i.e. sets \( M \) such that \( \mathcal{L}^*(M) = \{ 0, 1 \} \) (the two element Boolean algebra). Indeed the hyperhypersimple sets of Post [1944], although defined in terms of the intersection of arrays with the sets complement, also turned out to be related to this line of thought. Lachlan [1968] showed that they are precisely the r.e. sets \( A \) such that \( \mathcal{L}^*(A) \) is a Boolean algebra. He was also able to completely characterize the members of \( \mathcal{T} \) which are Boolean algebras as exactly the \( \Sigma_3 \) presentable ones.

At the other extreme one finds the \( r \)-maximal sets. These are easily seen to be equivalent to those with \( \mathcal{L}^*(A) \) having no complemented elements. Classifying the isomorphism types of the \( r \)-maximal sets however seems to be a difficult open problem. The only other commonly recognized principal filter in \( \mathcal{S}^* \) is the nearly ubiquitous one \( -\mathcal{S}^* \) itself. Of course if \( A \) is recursive it is immediate that \( \mathcal{L}^*(A) \cong \mathcal{S}^* \) but Soare [1974], [1981] has shown that this type is extremely common: If \( A \) is an r.e. infinite set and \( A \) is semilow (i.e. \( \{ e | W_e \cap A \neq \emptyset \} < \emptyset' \) then \( \mathcal{L}^*(A) \cong \mathcal{S}^* \). This means that there are r.e. sets \( A \) in every r.e. degree with \( \mathcal{L}^*(A) \cong \mathcal{S}^* \) and all low r.e. sets \( A \) (i.e., \( A' \leq_T \emptyset' \) have this property.

Our goal here is simply to provide some additional examples of types of principal filters in \( \mathcal{S}^* \). We will do this by describing some simple properties of the direct product of lattices in \( \mathcal{T} \). We will then use it to generate new isomorphism types in \( \mathcal{T} \). In addition these properties will enable us to make one really new identification. We will characterize the isomorphism type of \( \mathcal{L}^*(K) \) by an absorption property.
with respect to products in $\mathcal{L}$. Some connections between the structure of $\mathcal{L}^*(A)$ and the degree of $A$ will also be pointed out.

Our starting point is a simple fact from lattice theory. We work with distributive lattices with 0 and 1. Basic references are Birkhoff [1948] for lattice theory and Rogers [1967] for recursion theory. An excellent current survey of r.e. sets and degrees is Soare [1978].

**Lemma 1.** $L_1 \otimes L_2 \cong L$ iff there are $x_1$ and $x_2$ in $L$ such that $x_1 \wedge x_2 = 0$, $x_1 \vee x_2 = 1$ and $L_i \cong L(x_i)$ where $L(x_i) = \{ y \in L | y \geq x_i \}$.

**Proof.** The idea is just that $x_1, x_2$ are the images of $\langle 0, 1 \rangle$ and $\langle 1, 0 \rangle$ respectively. See Birkhoff [1948, p. 26].

Our first observation is that $\mathcal{S}$ is closed under products. Consider $\mathcal{L}^*(Ax)$ and $\mathcal{L}^*(A2)$. Let $R$ be an infinite co-finite recursive set with complement $\overline{R}$. We map \( f_1: \mathbb{N} \rightarrow R, f_2: \mathbb{N} \rightarrow \overline{R} \) by one-one onto recursive maps. It is then immediate that $\mathcal{L}^*(R \cup f_1[Ax]) \cong \mathcal{L}^*(Ax)$ and $\mathcal{L}^*(R \cup f_2[A2]) \cong \mathcal{L}^*(A2)$. Thus

$$\mathcal{L}^*(Ax) \otimes \mathcal{L}^*(A2) \cong \mathcal{L}^*(R \cup f_1[Ax]) \otimes \mathcal{L}^*(R \cup f_2[A2])$$

but by the lemma this is just $\mathcal{L}^*(f_1[Ax] \cup f_2[A2])$. Note that if $A_1$ and $A_2$ are simple so is $f_1[Ax] \cup f_2[A2]$. Thus the class of principal filters generated by simple sets is also closed under direct product. Of course $\mathcal{L}^*(\mathbb{N}) = \mathbb{1}$, the trivial one-element lattice, is an identity for products in $\mathcal{S}$.

We next consider $\mathcal{S}^*$ and see that it is an indecomposable idempotent.

**Corollary 2.** $\mathcal{S}^* \otimes \mathcal{S}^* \cong \mathcal{S}^*$.

**Proof.** Let $x_1$ and $x_2$ be given by any infinite co-finite recursive set and its complement.

**Corollary 3.** If $L_1 \otimes L_2 \cong \mathcal{S}^*$ then $L_i = \mathbb{1}$ or $\mathcal{S}^*$.

**Proof.** By the lemma the $L_i$ are isomorphic to $\mathcal{L}^*(A_i)$ for $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = \mathbb{N}$. Thus the $A_i$ are recursive and $\mathcal{L}^*(A_i) \cong \mathcal{S}^*$ or $\mathbb{1}$ (if one is $\mathbb{N}$).

Thus products of $\mathcal{S}^*$ give no new isomorphism types in $\mathcal{S}$ and of course products of Boolean algebras are still Boolean algebras. We can however combine these two known types to generate new ones. We use a more general version of Lemma 1 to prove that one gets new types in this way.

**Theorem 4.** If $\mathcal{S}^* \otimes L_1 \cong \mathcal{S}^* \otimes L_2$ then $L_1 \cong L_2$ or $L_1 = \mathcal{S}^* \otimes \mathbb{1}$ or $L_2 = \mathcal{S}^* \otimes L_1$.

**Proof.** By Theorem 7 on p. 26 of Birkhoff [1948] there are lattices $Z_1^1, Z_1^2, Z_2^1, Z_2^2$ such that $Z_1^1 \otimes Z_1^2 \cong \mathcal{S}^*$, $Z_1^1 \otimes Z_2^1 \cong \mathcal{S}^*$, $Z_1^1 \otimes Z_2^2 \cong \mathbb{1}$ and $Z_1^2 \otimes Z_2^2 \cong \mathbb{1}$. By Corollary 3, $Z_1^1, Z_1^2$ and $Z_2^1$ are $\mathcal{S}$ or $\mathcal{S}^*$. If $Z_1^1 \cong \mathbb{1}$ then $Z_1^2 = \mathcal{S}^*$ and so $L_1 \cong \mathcal{S}^* \otimes Z_2^2 \cong \mathbb{1}$ as required. Suppose now that $Z_1^1 \cong \mathcal{S}^*$. If $Z_1^2 \cong Z_2^2 = \mathbb{1}$ then again $L_1 \cong L_2 \cong Z_2^2$. If $Z_1^2 \cong Z_2^2$ then $Z_1^2 = \mathcal{S}^*$ and so $L_1 \cong \mathcal{S}^* \otimes Z_2^2$. The remaining cases are $(Z_1^1 = \mathcal{S}^* \& Z_1^2 = \mathcal{S}^*)$ and $(Z_1^1 \cong \mathcal{S}^* \& Z_1^2 \cong \mathbb{1})$. In the first case $L_1 \cong Z_2^2$ and $L_2 \cong \mathcal{S}^* \otimes Z_2^2$ as required. The second of course gives $L_2 \cong Z_2^2$ and $L_1 \cong \mathcal{S}^* \otimes Z_2^2$. □
Corollary 5. If $\mathfrak{B}_1 \cong \mathfrak{B}_2$ are Boolean algebras in $\mathcal{F}$ then $\mathfrak{L} \otimes \mathfrak{B}_1$ and $\mathfrak{L} \otimes \mathfrak{B}_2$ are nonisomorphic elements of $\mathcal{F}$. Neither is a Boolean algebra or $\mathfrak{L}$ but both are isomorphic to principal filters generated by simple sets.

Proof. As $\mathfrak{L}$ is not a Boolean algebra $\mathfrak{B}_i \cong \mathfrak{L} \otimes \mathfrak{B}_i$ and we apply the theorem. As there is a simple set $A$ with $\mathfrak{L}^*(A) \cong \mathfrak{L}$ the $\mathfrak{L} \otimes \mathfrak{B}_i$ are isomorphic to principal filters generated by simple sets by the closure of this class under direct product.

We next want to point out a simple relation between products of elements of $\mathcal{F}$ and the degrees of the r.e. sets to which they correspond.

Lemma 6. If $\mathfrak{L}^*(A) \cong \mathfrak{L}^*(A_1) \otimes \mathfrak{L}^*(A_2)$ then there are $B_i$ with $\mathfrak{L}^*(B_i) \cong \mathfrak{L}^*(A_i)$ and $B_1 \oplus B_2 \equiv_{\tau} A$.

Proof. The elements of $\mathfrak{L}^*(A)$ guaranteed by our basic lemma are now r.e. sets $B_1$ and $B_2$ with $B_1 \cap B_2 = A$, $B_1 \cup B_2 = \mathbb{N}$ and $\mathfrak{L}^*(B_i) \cong \mathfrak{L}^*(A_i)$. To see if $x \in B_i$ ask if $x \in A$. If so, $x \in B_i$. If not, enumerate both $B_1$ and $B_2$ until $x$ appears in one of them. If it first appears in $B_i$, $x \in B_i$ and otherwise $x \notin B_i$. Of course $x \in A$ iff $x \in B_1$ and $x \in B_2$.

Thus restrictions on $\mathfrak{L}^*(A)$ that push the degree of $A$ upward are passed on by products. So we have for example

Corollary 7. If $\mathfrak{L} \neq 1$ is a Boolean algebra and $\mathfrak{L}^*(A) \cong \mathfrak{L} \otimes \mathfrak{L}$ then $A$ is high i.e. $\mathfrak{L} = \mathfrak{L}^*$.

Proof. By Lachlan [1968] any $B$ with $\mathfrak{L}^*(B) = \mathfrak{L}$ is hyperhypersimple. Martin [1966] then shows that $B$ must be high.

Carrying this idea to an extreme one might guess that the most complicated sets $A$ should have $\mathfrak{L}^*(A)$’s with the most factors. Indeed this gives us our characterization of $\mathfrak{L}^*(K)$. ($K = \{e | e \in \mathbb{W}_e\}$ is of course a 1-complete r.e. set and so in many ways the most complicated one.)

Theorem 8. $\mathfrak{L}^*(K) \cong \mathfrak{L}^*(K) \otimes \mathfrak{L}^*(A)$ for every r.e. $A$.

Proof. Let $R_1, f_1$ and $f_2$ be as in the proof that $\mathcal{F}$ is closed under products following Lemma 1. As before we see that $\mathfrak{L}^*(K) \otimes \mathfrak{L}^*(A) \cong \mathfrak{L}^*(f_1[K] \cup f_2[A])$. As $f_1$ is a recursive one-one and onto map of $\mathbb{N}$ to a recursive set $R$, $f_1[K]$ is also a complete set as is $f_1[K] \cup f_2[A]$. Thus by Myhill [1955] $K$ and $f_1[K] \cup f_2[A]$ are recursively isomorphic and so

$$\mathfrak{L}^*(K) \cong \mathfrak{L}^*(f_1[K] \cup f_2[A]) \cong \mathfrak{L}^*(K) \otimes \mathfrak{L}^*(A)$$

as required. □

This theorem characterizes the isomorphism type of $\mathfrak{L}^*(K)$ for if $\mathfrak{L}^*(B) \cong \mathfrak{L}^*(B) \otimes \mathfrak{L}^*(A)$ for every $A$ then $\mathfrak{L}^*(B) \cong \mathfrak{L}^*(B) \otimes \mathfrak{L}^*(K) \cong \mathfrak{L}^*(K)$. Moreover our earlier results show that the type of $\mathfrak{L}^*(K)$ is not any of the ones considered before, i.e., it is not generated as a product of lattices which are Boolean algebras or $\mathfrak{L}$.

Finally our results on degrees show that if $\mathfrak{L}^*(K) \cong \mathfrak{L}^*(A)$ then $A$ is high.
Bibliography


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