SHAPE EQUIVALENCE DOES NOT IMPLY CE EQUIVALENCE

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Abstract. We give an example of shape equivalent compacta $X$ and $Y$ such that there is no compactum $Z$ with cell-like maps $Z \to X$ and $Z \to Y$.

A space $X$ is said to be cell-like if for some imbedding of $X$ in an ANR, $X$ has the property that for each neighborhood $U$ of $X$, $X$ contracts to a point in $U$. This is an intrinsic property of $X$ and is independent of the choice of ANR and embedding. A continuous map $f: Z \to Y$ between compacta is said to be cell-like (CE) if $f$ is surjective and $f^{-1}(y)$ is cell-like for each $y \in Y$. If $X$ and $Y$ are compacta, we say that $X$ and $Y$ are CE equivalent if there exist compacta $X = X_0, X_1, \ldots, X_{2k} = Y$ and CE maps $f_{2i}: X_{2i+1} \to X_{2i}$ and $f_{2i+1}: X_{2i+1} \to X_{2i+2}$ for $i = 0, 1, \ldots, k - 1$.

In [F1] it is shown that two compacta which are homotopy equivalent must be CE equivalent. In fact, more is shown. The maps constructed have sections and contractible point-inverses. It is natural to seek a Čech analog of this theorem for general compacta. Thus, we are led to study the question: "If $X$ and $Y$ are shape equivalent compacta, must $X$ and $Y$ be CE equivalent?"

In this note we will exhibit a simple example which shows that this is not the case. Let $X$ be a plane compactum which is the union of a circle $C$ and a ray $R$.
which spirals into C. See Figure 1. X is shape equivalent to $S^1$. We will show that X
is not CE equivalent to $S^1$.

**Definition 1.** We will say that a compactum $Z$ is an acyclic image if there exist a
compactum $W$ with $\tilde{H}^*(W) = \tilde{H}^*(\text{pt})$ and a continuous surjection $f: W \to Z$.

**Lemma 1.** Let $P$ and $Q$ be CE equivalent compacta. Then $P$ is an acyclic image if
and only if $Q$ is an acyclic image.

**Proof.** It suffices to consider the case in which there is a CE map $r: P \to Q$. It is
clear that $Q$ is an acyclic image if $P$ is an acyclic image. Suppose, then, that $Q$ is an
acyclic image. Let $f: W \to Q$ be a surjection as in Definition 1 and let $E$ be the
pullback in the diagram below.

$$
\begin{array}{ccc}
E & \xrightarrow{j} & P \\
\downarrow \text{CE} & & \downarrow \text{CE} \\
W & \xrightarrow{f} & Q
\end{array}
$$

$E$ is compact, $\tilde{j}$ is surjective, and $\tilde{r}$ is CE. A cell-like set has the Čech cohomology
of a point, so the Vietoris-Begle theorem [S] implies that $\tilde{r}$ induces an isomorphism
of Čech cohomology. Thus, $E$ has the Čech cohomology of a point and $P$ is an
acyclic image. □

**Lemma 2.** The space $X$ of Figure 1 is not an acyclic image.

**Proof.** Suppose not. Let $f: W \to X$ be a surjection as in Definition 1. Let $r: X \to C$ be a radial retraction and let $e: E^1 \to C$ be the universal cover. Since
$\tilde{H}^1(W) \cong [W, C] = 0$, the composition $r \circ f: W \to C$ lifts to $E^1$ and there is a map
$\tilde{f}: W \to E^1$ such that $e \circ \tilde{f} = r \circ f$.

Let $W' = f^{-1}(R)$. Choose a map $\tilde{r}: R \to E^1$ so that $e \circ \tilde{r} = r|_R$ and so that
$\tilde{r} \circ f = \tilde{f}$ for some point $w_0 \in W'$. Let $W'' = \{w \in W' | \tilde{r} \circ f(w) = \tilde{f}(w)\}$. The usual argument shows that $W''$ is open in $W'$ and therefore in $W$. $W''$ cannot be closed in $W$ since $W$ is connected and $W''$ is neither empty nor all of $W$.

There is therefore a sequence $\{w_i\} \subseteq W''$ converging to a point $w^* \in W - W''$. Thus, $\lim f(w_i) \in C$ and $\{\tilde{r} \circ f(w_i)\}$ is unbounded in $E^1$. On the other hand, $\{\tilde{r} \circ f(w_i)\} = \{\tilde{f}(w_i)\} \subseteq \tilde{f}(W')$, which is compact. This is the desired contradiction. □

This completes the proof of our main result, since there is a continuous map of
$[0, 1]$ onto $S^1$.

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2 Lemma 2 is essentially Theorem 1 of M. K. Fort [F0].

It would be interesting to find shape equivalent UV$^1$ compacta which are not CE equivalent. Parts of
[F3] are relevant to this problem.
References


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