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WEYL GROUP ACTIONS AND EQUIVARIANT HOMOTOPY EQUIVALENCE

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ABSTRACT. Let G be a compact Lie group and G_0 its identity component. Then we shall show that the normal representations of the corresponding fixed point components of G-homotopy equivalent manifolds are necessarily isomorphic when G/G_0 is a Weyl group of a compact connected Lie group.

1. Introduction. In [7] and [8], we defined an equivariant J-homomorphism J_G: $KO_G(X) \to J_G(X)$ for a compact G-space X and showed an equivariant analogue of Atiyah [2]. A corollary peculiar to the equivariant case is the following. The normal representations of the corresponding fixed point components of G-homotopy equivalent manifolds are stably homotopy equivalent.

In the present paper, we study the equivariant J-homomorphism in the case where X is a point $*$. Let G be a compact Lie group and G_0 its identity component. Then we shall show that J_G: $RO(G)$ $(\cong KO_G(*)) \to J_G(*)$ is an isomorphism if G/G_0 is (for example) a Weyl group of a compact connected Lie group (Theorem 4.2).

By combining the above results, we have strong necessary conditions for two G-manifolds to be G-homotopy equivalent.

Namely the normal representations of the corresponding fixed point components of G-homotopy equivalent manifolds have to be isomorphic if G/G_0 is a Weyl group (Theorem 5.4).

Note that all the symmetric groups S_n are realized as the Weyl groups of the unitary groups $U(n)$. Consequently J_{S_n}: $RO(S_n) \to J_{S_n}(*)$ is an isomorphism.

However J_G: $RO(G) \to J_G(*)$ is not an isomorphism in general for G an alternating group A_n. We shall deal with J_{A_n} in a forthcoming paper.

In [13], R. Schultz studied the relation between linear equivalence and topological equivalence of group representations. In §6, we study the relations among linear equivalence, topological equivalence and stable homotopy equivalence of group representations.

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2. Equivariant J-homomorphism. Let G be a compact topological group and X be a compact G-space. Let ξ and η be G-vector bundles over X. Denote by $S(\xi)$ (resp. $S(\eta)$) the sphere bundle associated with ξ (resp. η).

DEFINITION 2.1. $S(\xi)$ and $S(\eta)$ are said to be of the same G-fiber homotopy type if there exist fiber-preserving G-maps

$$f\colon S(\xi) \to S(\eta), \qquad f'\colon S(\eta) \to S(\xi),$$

and fiber-preserving G-homotopies,

$$h\colon S(\xi) \times I \to S(\xi), \qquad h'\colon S(\eta) \times I \to S(\eta),$$

with

$$h|S(\xi) \times 0 = f' \cdot f, \quad h|S(\xi) \times 1 = \text{identity},$$
$$h'|S(\eta) \times 0 = f \cdot f', \quad h'|S(\eta) \times 1 = \text{identity}.$$

Let $KO_G(X)$ be the Grothendieck-Atiyah-Segal group [3] defined in terms of real G-vector bundles over X. Let $T_G(X)$ be the additive subgroup of $KO_G(X)$ generated by elements of the form $[\xi] - [\eta]$, where ξ and η are G-vector bundles whose associated sphere bundles are G-fiber homotopy equivalent.

DEFINITION 2.2. We define our equivariant J-group $J_G(X)$ by $J_G(X) = KO_G(X)/T_G(X)$ and define our equivariant J-homomorphism J_G by the natural epimorphism $J_G\colon KO_G(X) \to J_G(X)$.

Our equivariant J-homomorphism is an equivariant version of the J-homomorphism of Adams [1]. Precisely speaking, $J_G(X) \cong J(X)$ and $J_G\colon KO_G(X) \to J_G(X)$ coincides with $J\colon KO(X) \to J(X)$ of Adams [1] when G is the trivial group.

3. Equivariant Adams operations. In this section we recall some results in [4] and [12]. Denote by Q the rational number field. Let G be a finite group of order N and let Γ_N be the Galois group of $Q(\omega)$ over Q where ω is a primitive Nth root of unity. It is well known that Γ_N is isomorphic to the multiplicative group Z_N^* of units in the ring Z_N. Denote by $RO(G)$ the real (orthogonal) representation ring of G. Then Γ_N acts on $RO(G)$ in two manners. For $s \in Z_N^*$ ($\cong \Gamma_N$) and for a G-representation V, the equivariant Adams operation is defined by

$$\Psi^s(V) = P_s(\Lambda^1 V, \Lambda^2 V, \dots, \Lambda^n V)$$

where $P_s(\sigma_1, \dots, \sigma_n) = x_1^s + \cdots + x_n^s$, σ_i the ith elementary symmetric function of $x_1, \dots, x_n$, $n > \dim V$, and $\Lambda^i V$ is the ith exterior power of V.

On the other hand, it is well known (see for example [6]) that V is realized by a homomorphism $\rho\colon G \to \mathrm{GL}(\dim V, Q(\omega))$. For $s \in Z_N^*$, let α^s be the element of the Galois group Γ_N defined by $\alpha^s(\omega) = \omega^s$. Then $\Psi^s V$ is associated to the homomorphism $\bar{\alpha}^s \circ \rho$ where $\bar{\alpha}^s\colon \mathrm{GL}(\dim V, Q(\omega)) \to \mathrm{GL}(\dim V, Q(\omega))$ is induced by α^s. It is shown in [4] and [12] that the above two definitions of Ψ^s coincide.

DEFINITION 3.1. Let $WO(G)$ be the subgroup of $RO(G)$ generated by the $V - \Psi^s V$ where $V \in RO(G)$, $s \in Z_N^*$.

DEFINITION 3.2. Let $WO(G)' = \{V - W \in RO(G) | \dim V^H = \dim W^H$ for all subgroups H of $G\}$ where $V^H = \{v \in V | hv = v$ for all $h \in H\}$ and similar for W^H.

Then Lee and Wasserman showed the following proposition on which our results will be based.

PROPOSITION 3.3 (PROPOSITION 3.17 OF **[12]**). *For any finite group, $WO(G) = WO(G)'$.*

4. Groups G for which J_G is an isomorphism. In this section, we consider J_G: $KO_G(X) \to J_G(X)$ in the case where X is a point $*$. Let G be a compact Lie group and G_0 its identity component.

LEMMA 4.1. *If all real representations of G/G_0 have rational characters, then J_G: $RO(G)$ $(\cong KO_G(*)) \to J_G(*)$ is an isomorphism.*

PROOF. We show that $T_G(*) = \{0\}$. Let $V - W$ be an arbitrary element of $T_G(*)$. We may suppose that the unit spheres $S(V)$ and $S(W)$ are G-homotopy equivalent. Denote by V^{G_0} (resp. W^{G_0}) the fixed point set of G_0 and by V_{G_0} (resp. W_{G_0}) its orthogonal complement in V (resp. W). According to **[16]**, $S(V)$ and $S(W)$ are G-homotopy equivalent if and only if the following two conditions hold.

(1) $S(V^{G_0})$ and $S(W^{G_0})$ are G/G_0-homotopy equivalent.

(2) V_{G_0} and W_{G_0} are isomorphic G-representations.

Hence $V^{G_0} - W^{G_0}$ belongs to $T_{G/G_0}(*)$. In particular, $\dim S(V^{G_0})^H = \dim S(W^{G_0})^H$ for all subgroups H of G/G_0. Namely $V^{G_0} - W^{G_0}$ belongs to $WO(G/G_0)'$. It follows from Proposition 3.3 that $V^{G_0} - W^{G_0}$ also belongs to $WO(G/G_0)$. It is Galois theory that the elements of $RO(G/G_0)$ invariant under Γ_N consist precisely of those virtual representations with rational characters where N is the order of the group G/G_0. Therefore $WO(G/G_0) = \{0\}$ by the assumption of Lemma 4.1. It follows that $V^{G_0} - W^{G_0} = 0$ in $RO(G/G_0)$. Accordingly $V^{G_0} - W^{G_0} = 0$ in $RO(G)$.

On the other hand $V_{G_0} - W_{G_0} = 0$ in $RO(G)$ (see (2) above). Thus we have that $V - W = 0$ in $RO(G)$.

This completes the proof of Lemma 4.1.

THEOREM 4.2. *If G/G_0 is a Weyl group of a compact connected Lie group, then J_G: $RO(G) \to J_G(*)$ is an isomorphism.*

PROOF. The characters of complex (unitary) representations of a Weyl group of a simple Lie group are rational valued. This fact was shown by A. Young **[17]** for the families of groups of type A_l and B_l, by W. Specht **[15]** for the family of groups of type D_l, by T. Kondo **[11]** for the group of type F_4 and by M. Benard **[5]** for the groups of type E_6, E_7 and E_8. This result is easily shown for the group of type G_2, which is dihedral of order 12. In fact, they showed stronger results as follows. Any irreducible character of the above groups has Schur index 1 over Q, i.e., any complex representation of the above groups is a rational representation.

It is easy to see that a Weyl group of a compact connected Lie group is isomorphic to the direct product of Weyl groups of several simple Lie groups. Note that an irreducible complex representation of the direct product $G_1 \times G_2$ of finite groups G_1, G_2 is the tensor product of some irreducible representations of G_1 and G_2 (see for example **[14]**). Hence a complex representation of $G_1 \times G_2$ is rational valued if all irreducible representations of G_1 and G_2 are rational valued.

Thus we have that the characters of complex representations of a Weyl group of a compact connected Lie group are rational valued. If χ is the character of a real representation V, then the complexification $cV = \mathbf{C} \otimes_R V$ has the same character χ. This implies the character of a real representation of G/G_0 is also rational valued. Hence Theorem 4.2 follows from Lemma 4.1.

5. A necessary condition for G-homotopy equivalence. We first recall the following

THEOREM 5.1 ([7], [8]). *Let G be a compact Lie group and M_1, M_2 be closed smooth G-manifolds with tangent bundles TM_1, TM_2 respectively. Let f: $M_1 \to M_2$ be a G-homotopy equivalence. Then we have $J_G([TM_1]) = J_G([f^* TM_2])$.*

Let f: $M_1 \to M_2$ be a G-homotopy equivalence. Denote by F_1^μ each component of the fixed point set of M_1. Set $F_2^\mu = f(F_1^\mu)$. Then F_2^μ is a component of the fixed point set of M_2 and the union $\bigcup_\mu F_2^\mu$ is exactly the fixed point set of M_2. Denote by N_i^μ the normal bundles of F_i^μ in M_i $(i = 1, 2)$. Then we have

COROLLARY 5.2. $J_G([N_1^\mu]) = J_G([(f|F_1^\mu)^* N_2^\mu])$.

Let V_i^μ be the normal representations of F_i^μ in M_i $(i = 1, 2)$. Regarding V_i^μ as G-vector bundles over one point, we have in particular

COROLLARY 5.3. $J_G([V_1^\mu]) = J_G([V_2^\mu])$.

By combining Theorem 4.2 and Corollary 5.3, we obtain a necessary condition for two G-manifolds to be G-homotopy equivalent.

THEOREM 5.4. *If G/G_0 is a Weyl group of a compact connected Lie group, then V_1^μ and V_2^μ are isomorphic G-representations.*

6. Relations among linear equivalence, topological equivalence and stable homotopy equivalence. In [13], R. Schultz gave some classes of groups for which linear equivalence equals topological equivalence. Among the classes of Corollary 2.3 in [13], even stable homotopy equivalence equals linear equivalence [9] for (i) and (iv) where

(i) all groups of the form $(Z_2)^k$, $k \geqslant 1$,

(iv) all groups of the form $Z_4 \times (Z_2)^k$.

On the contrary, homotopy equivalence does not equal linear equivalence for all cyclic groups Z_n with $n \neq 1, 2, 3, 4, 6$ [9].

More generally, stable homotopy equivalence equals linear equivalence for a compact Lie group G if $G/G_0 \cong$ a Weyl group or if each cyclic subgroup of G/G_0 is of order 1, 2, 3, 4 or 6 [10]. Examples are the following classes of compact Lie groups G.

(i) $G/G_0 \cong (S_n)^k$, $(A_4)^k$, $(D_3)^k$, $(D_4)^k$ where $n, k \geqslant 1$ and D_n are dihedral groups.

(ii) $G/G_0 \cong (Z_2)^j \times (Z_3)^k$, $(Z_2)^j \times (Z_4)^k$, where $j, k \geqslant 0$.

(iii) $G \cong \prod_i O(a_i) \times \prod_j \mathrm{Pin}(b_j) \times \prod_k \mathrm{Pin}^c(c_k) \times \prod_m NU(d_m)$,

where $NU(n)$ denotes the group consisting of all unitary or conjugate-linear unitary maps of $\mathbf{C}^n$ and a_i, b_j, c_k and d_m are arbitrary nonnegative integers.

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