A DIRECT PROOF THAT THE RC-INTEGRAL IS EQUIVALENT
TO THE D*-INTEGRAL

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ABSTRACT. We give a direct and short proof that the Riemann complete integral
defined by R. Henstock is equivalent to the special Denjoy integral.

1. Introduction. It is well known that the Riemann complete integral (RC-integral) defined by R. Henstock in [4] is equivalent to the special Denjoy integral (D*-integral). This fact is proved by the following procedure. The RC-integral is equivalent to the variational integral [3] which is also equivalent to the Ward integral [2]. The Ward integral is equivalent to the Perron integral [6] which is, in turn, identical with the D*-integral by the Theorem of Hake-Alexandroff-Looman [5].

The object of this paper is to give a direct and short proof of this equivalence. For fundamental terminologies and notations on the RC- and D*-integrals, we refer to [3] and [5] respectively.

If $J = [\alpha, \beta]$, then J^0 is the open interval (α, β).

2. The Riemann complete integral. We begin by reproducing the definition of the RC-integral for completeness [4].

Let f be a real-valued function defined on a compact interval $I_0 = [a, b]$. Then f is said to be RC-integrable on $[a, b]$ to the value $A = (RC)\int_a^b f$ if, for any $\varepsilon > 0$, there exists a positive real-valued function $\delta(x)$ on $[a, b]$ such that whenever $a = x_0 < x_1 < \cdots < x_n = b$ and $x_{i-1} \leqslant \xi_i \leqslant x_i$, $x_i - x_{i-1} < \delta(\xi_i)$ $(i = 1, 2, \ldots, n)$, we have

$$\left| \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) - A \right| < \varepsilon.$$

We state some important properties of the RC-integral which will be used in what follows.

A finite function F is said to be AC* on E, if F is bounded on an interval containing E and if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that, for every finite sequence of nonoverlapping intervals $\{I_k\}$ whose end points belong to E, the inequality $\sum_k |I_k| < \delta$ implies $\sum_k O(F, I_k) < \varepsilon$, where $O(F, I_k)$ means the oscillation of F on I_k. A function will be termed ACG* on E, if the function is continuous on E and if E is expressible as the sum of a sequence of sets on each of which the function is AC*.

PROPERTY 1 ([3, THEOREMS 30.3, 32.1 AND 35.1]). If f is RC-integrable on I_0 then its indefinite integral $F(x)$ is ACG* on I_0 and $F'(x) = f(x)$ a.e.

PROPERTY 2 ([1] OR [4]). If f is L-integrable on I_0, then f is also RC-integrable on I_0 and $(RC)\int_{I_0} f = (L)\int_{I_0} f$.

PROPERTY 3 ([3, THEOREM 46.1]). If f is RC-integrable on every interval $[a, \beta]$ $(a < \beta < b)$ and if $\lim_{\beta \to b}(RC)\int_a^\beta f = A$, then f is RC-integrable on $[a, b]$ and $(RC)\int_a^b f = A$.

PROPERTY 4 ([3, THEOREM 48.1]). If f is L-integrable on a closed set E with end points a, b and complementary intervals $\{(a_n, b_n)\}$, while f is RC-integrable on each interval $I_n = [a_n, b_n]$, and if $\sum_n O(F_n, I_n) < \infty$, where F_n is an indefinite integral of f on I_n, then f is RC-integrable on $[a, b]$ and

$$(RC)\int_a^b f = (L)\int_E f + \sum_{n=1}^\infty (RC)\int_{I_n} f.$$

3. Proof of the equivalence.

THEOREM 1. If f is RC-integrable on $[a, b]$ then for any closed set $E \subset [a, b]$, there exists a closed interval J with $J^0 \cap E \neq \emptyset$ such that (i) f is L-integrable on $J \cap E$, (ii) if $\{J_k = [a_k, b_k]\}$ are the intervals contiguous to $J \cap E$ with respect to J, then $\sum_{k=1}^\infty O(F, J_k) < \infty$, where F is an indefinite integral of f.

PROOF. By Property 1, $F(x)$ is ACG* on $[a, b]$ and $F'(x) = f(x)$ a.e. Since F is ACG*, there exists a closed interval J with $J^0 \cap E \neq \emptyset$ such that F is AC* on $J^0 \cap E$ [5, p. 233]. Therefore F is AC* on its closure $\overline{J^0 \cap E}$ by the continuity of F. An end point of J may be $J \cap E$ without being in $\overline{J^0 \cap E}$; but if this is so, then the end point is an isolated point of $J \cap E$. Hence F is AC* on $J \cap E$, and is VB* [5, p. 228] on $J \cap E$. It follows from the definition of VB* that $\sum_k O(F, J_k) < \infty$.

We denote by $G(x)$ the function which coincides with $F(x)$ on $J \cap E$ and is linear on each J_k. Then the function G is AC* on J and hence G' is L-integrable on J. Since $G'(x) = F'(x) = f(x)$ at almost all points of $J \cap E$, f is L-integrable on $J \cap E$.

REMARK 1. By the definition of the D*-integral [5, p. 241], we see that this proof is valid for the D*-integral. Hence Theorem 1 is true for the D*-integral.

REMARK 2. If we take a closed interval as E in Theorem 1, then we have the following proposition. If f is RC-integrable on $[a, b]$ then there exists a closed subinterval J on which f is L-integrable.

THEOREM 2. The RC-integral is equivalent to the D*-integral.

PROOF. Throughout the proof, I, J denote closed intervals.

Let f be an RC-integrable function on $I_0 = [a, b]$. Then, by Property 1, its indefinite integral F is ACG* on I_0 and $F'(x) = f(x)$ a.e. Hence, by the definition of the D*-integral, f is D*-integrable and

$$(D^*)\int_a^b f = (RC)\int_a^b f.$$

To prove the converse, let f be D*-integrable on I_0 and I be a subinterval of I_0. Then f is allowed to take infinite values at the points of the set with measure zero. However these infinite values may be replaced by zero without altering the integral. Hence we can assume that f is finite at every point.

If f is RC-integrable on I and $(RC)\int_I f = (D^*)\int_I f$, then we call the interval I regular. By Remark 2, there exists an interval J on which f is L-integrable. Since the RC-integral includes the L-integral (Property 2), J is a regular interval. If x_0 is an interior point of some regular interval, then x_0 is called a regular point. Otherwise we call the point irregular.

(i) First we shall show that if the interval I has no irregular points in its interior then any interval $J \subset I^0$ is regular.

Suppose that there exists an interval J which is not regular. If we divide J into two intervals by its middle point then at least one of these intervals is not regular since the integrals RC and D* both have the additivity of the intervals. Continuing this process indefinitely, we have a sequence of intervals $\{J_k\}$ which are not regular. Let x_0 be the common point of the J_k. Then x_0 is an irregular point contained in I^0, which contradicts the hypothesis.

(ii) Next we shall show that if any interval $J \subset I^0$ is regular, then I itself is regular.

We put $I = [\alpha, \beta]$ and $J = [c, d]$ where $\alpha < c < d < \beta$. Since J is regular, we have, by definition, $(RC)\int_c^d f = (D^*)\int_c^d f$. It follows from the Cauchy property of D*-integral [5, p. 247] that

$$\lim_{\substack{c \to \alpha \\ d \to \beta}} (RC) \int_c^d f = (D^*) \int_\alpha^\beta f.$$

Hence, by Property 3, f is RC-integrable on I, and $(RC)\int_\alpha^\beta f = (D^*)\int_\alpha^\beta f$.

(iii) We have to show that I_0 is a regular interval. Suppose that I_0 is not regular. Then, by (i) and (ii), I_0 contains at least one irregular point. Let E be the set of all irregular points. Then E is closed. It follows from Remark 1 that there exists an interval J with $J^0 \cap E \neq \varnothing$ such that f is L-integrable on $J \cap E$ and if $\{J_k = [a_k, b_k]\}$ is the set of contiguous intervals of $J \cap E$ with respect to J, then $\sum_k O(D^*, f, J_k) < \infty$ where $O(D^*, f, J_k)$ means the oscillation of indefinite D*-integral on J_k. Hence, by the Harnack property of the D*-integral [5, p. 257], we have

$$(D^*) \int_J f = (L) \int_{J \cap E} f + \sum_{k=1}^\infty (D^*) \int_{J_k} f.$$

Since $J_k^0 \cap E \neq \varnothing$, it follows from (i) and (ii) that each I_k is regular and that

$$(D^*) \int_{J_k} f = (RC) \int_{J_k} f,$$

$$\sum_{k=1}^\infty O(D^*, f, J_k) = \sum_{k=1}^\infty O(F_k, J_k) < \infty,$$

where F_k is an indefinite RC-integral on J_k. Hence we have from Property 4 that

$$(D^*)\int_J f = (L)\int_{J \cap E} f + \sum_{k=1}^{\infty} (D^*)\int_{J_k} f$$

$$= (L)\int_{J \cap E} f + \sum_{k=1}^{\infty} (RC)\int_{J_k} f = (RC)\int_J f.$$

Therefore J is regular. But this contradicts the relation $J^0 \cap E \neq \varnothing$. Hence I_0 must be regular, which completes the proof.

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