HOMOTOPY RIGIDITY FOR GRASSMANNIANS

ALLEN BACK

Abstract. Two n-dimensional unitary representations which differ by complex conjugation or tensoring with a character induce topologically equivalent actions on the Grassmann manifold of complex m-planes in n-space. This paper shows under modest dimension hypotheses that only such projectively equivalent linear representations of compact connected Lie groups can give topologically conjugate actions.

A linear representation \( \alpha: G \to U(N) \) induces a \( G \)-action on the Grassmann manifold \( G_m(C^N) = U(N)/U(m) \times U(N-m) \) of \( m \)-planes in complex \( N \)-space. If \( \bar{\alpha} \) is the conjugate of \( \alpha \) and \( \chi \) is any character of \( G \), it is obvious that \( \alpha \otimes \chi \) and \( \bar{\alpha} \otimes \chi \) give rise to the same actions. The purpose of this paper is to show for most values of \( m \) and \( N \) that only such obviously equivalent linear actions give topologically conjugate actions on \( G_m(C^N) \). The main result is as follows.

Theorem 1. Let \( \alpha \) and \( \beta \) be \( N \)-dimensional unitary representations of a compact connected Lie group \( G \). Suppose that there is a homotopy equivalence \( f: G_m(C^N) \to G_m(C^N) \) \((N > m)\) which is a \( G \)-map between \( G_m(C^N) \) with the two linear actions \( \alpha \) and \( \beta \). Further suppose that either

(a) \( N \neq 2m \), or
(b) \( m < 3 \).

Then there is a character \( \chi: G \to S^1 \) so that \( \beta \) is equivalent as a linear representation to either \( \alpha \otimes \chi \) or \( \bar{\alpha} \otimes \chi \).

In the language of Arunas Liulevicius ([1], [2], [3], [4]), this theorem says that compact connected Lie group actions on \( G_m(C^N) \) satisfy the property of homotopy rigidity. Wu-Yi Hsiang observed that such results could sometimes be proved efficiently by the use of equivariant cohomology (see e.g. [5]). We shall use this technique to prove Theorem 1. The author would like to express his appreciation to Wu-Yi Hsiang, Arunas Liulevicius, and Birgit Speh for helpful comments.

Because equivariant cohomology is invariant under \( G \)-maps which are homotopy equivalences, our strategy for proving Theorem 1 is quite straightforward. We shall simply show that two linear actions which are not obviously equivalent will in fact have nonisomorphic equivariant cohomologies.

The dimension restrictions in Theorem 1 arise completely from uncertainty about the group of automorphisms of the integral cohomology of the Grassman-
nian in this case. It is a very viable conjecture that only the obvious automorphisms occur.

In order to study the equivariant cohomology, we shall recall some results about the cohomology of the Grassmannian (see e. g. [6]). As is well known, the cohomology of the infinite Grassmannian of \( m \)-planes may be identified with the algebra of symmetric functions in \( m \)-variables \( t_1, t_2, \ldots, t_m \). The elementary symmetric functions \( \sigma_1, \sigma_2, \ldots, \sigma_m \) are then the Chern classes of the canonical \( m \)-plane bundle. Define classes \( h_k \in H^{2k}(BU(m); Z) \subseteq Z[t_1, \ldots, t_m] \) inductively by

\[
h_0 = 1 \quad (h_i = 0 \text{ for } i < 0),
\]

\[
h_{n+m} = \sum_{j=1}^{m} (-1)^{j-1} \sigma_j h_{n+m-j}. \quad (*)
\]

Let \( \langle h_{N-m+1}, h_{N-m+2}, \ldots, h_N \rangle \) be the ideal in \( Z[h_1, \ldots, h_m] = Z[\sigma_1, \ldots, \sigma_m] \) generated by \( h_{N-m+1}, \ldots, h_N \).

**Theorem 2.**

1. \( H^*(G_m(C^N); Z) = Z[h_1, \ldots, h_m]/\langle h_{N-m+1}, \ldots, h_N \rangle \).
2. A module basis for \( H^*(G_m(C^N); Z) \) is given by all \( m \)-fold products

\[
h_{i_1} \cdot n \cdot h_{i_m} \quad \text{with } 0 < i_1 < i_2 < \cdots < i_m < N-m.
\]

The \( h_k \) are the so-called "complete symmetric functions." They may be identified (up to sign) with the Chern classes of the canonical \( N-m \)-plane bundle. Their generating function \( H = \sum_{k=0}^{\infty} h_k \) is given by

\[
\prod_{j=1}^{m} \left( \frac{1}{1 - t_j} \right) = 1/\left( \sum_{j=0}^{m} (-1)^{j-1} \sigma_j \right),
\]

and hence \( h_k \) can also be described as the unique degree \( k \) symmetric polynomial each of whose monomials \( t_1^{i_1} t_2^{i_2} \cdots t_m^{i_m} \) has coefficient 1.

For compact connected Lie groups, similarity of linear representations is entirely detected by restriction to a maximal torus. Also any character of the maximal torus which is invariant under the Weyl group action will be the restriction of a character of the group. Hence it suffices to prove Theorem 1 when \( G \) is a torus \( T \). Let \( R = H^*(BT; Z) = Z[x_1, \ldots, x_p] \) where \( p \) is the rank of the torus. Note that by transgression, the weights of a representation \( \alpha: T \to U(N) \) may be viewed as two dimensional elements of \( R \).

Recall that the equivariant cohomology \( H_T^*(X) \) of a \( T \)-space \( X \) is defined to be the ordinary cohomology of the space \( X_T \), formed by the Borel construction. More precisely, \( X_T \) is the total space of the bundle

\[
X \to X_T = ET \times_T X
\]

\[
\downarrow
\]

\[
BT
\]

which is associated to the universal \( T \)-bundle by the \( T \)-action on \( X \).

Now in the case where \( X = G_m(C^N) \) with a linear \( T \)-action, the canonical bundles \( \gamma^m \) and \( \gamma^{N-m} \) are equivariant bundles. Hence the Borel construction may be applied to them to give bundles \( (\gamma^m)_T \) and \( (\gamma^{N-m})_T \) over \( X_T \). Inasmuch as the
Chern classes $c_i(\gamma^m) = \sigma_i$ generate $H^*(X; \mathbb{Z})$, we see that the Chern classes of $(\gamma^m)_T$ provide explicit lifts of $H^*(X; \mathbb{Z})$ to $H^*_T(X; \mathbb{Z})$. Hence the fiber of the Borel fibration is totally nonhomologous to zero. Also for the trivial bundle of dimension $N$ with nontrivial torus action given by $\alpha$, it is easy to see that $c_k(\xi_T)$ is the $k$th elementary symmetric function $\rho_k$ of the weights $w_i$ of $\alpha$. As $(\gamma^m)_T \otimes (\gamma^{N-m})_T = \xi_T$, we may use the above observations to obtain the following (see [6] for more detail).

**Theorem 3.** Let $T$ act on $G_m(C^N)$ by a linear representation $\alpha$ with weights $w_i$. Then

$$H^*_T(G_m(C^N); \mathbb{Z}) = R[ h_1, \ldots, h_m]/\langle f_{N-m+1}, \ldots, f_N \rangle$$

where

$$f_m = \sum_{j=0}^{M} (-1)^j \rho_j h_{m-j}$$

and $\rho_j$ is the $j$th elementary symmetric function of the weights $w_i$.

It is worth noticing that the classes $h_k \in H^*_T(G_m(C^N); \mathbb{Z})$ are defined by the recursion formulas (*) where the $\sigma_i$ are now taken to be the Chern classes $c_i((\gamma^m)_T)$. The bundle $\gamma^m$ with action $\alpha \otimes \chi$ is $T$-isomorphic to the tensor product of $\gamma^m$ with action $\alpha$ and a 1-dimensional trivial bundle with action $\chi$. Let $w$ be the weight of $\chi$. Then replacement of $\alpha$ by $\alpha \otimes \chi$ replaces the elementary symmetric function $\sigma_k$ of $t_1, \ldots, t_m$ by the corresponding symmetric function of $t_1 + w, \ldots, t_m + w$.

By restriction to the fiber in the Borel bundle, it is clear that any isomorphism of equivariant cohomologies which is induced by a $T$-map will in turn induce an automorphism of $H^*(G_m(C^N); \mathbb{Z})$. The following has been shown by Brewster [9] (see also [8] and [7] for special cases with simpler proofs).

**Theorem 4.** Suppose $N > 3$. If $N \neq 2m$, the only nontrivial automorphism of $H^*(G_m(C^N); \mathbb{Z})$ is the involution induced by conjugation $c$ of complex numbers. Explicitly $c^*(h_k) = (-1)^k h_k$.

The above result is a special case of [2, Conjecture 5].

**Conjecture.** If $N = 2m$, $m > 1$ then the group of automorphisms is $Z/2 \times Z/2$ generated by conjugation and the map $d$ taking an $m$-plane to the $m$-plane orthogonal to it. Explicitly $d^*(h_m) = (-1)^m \sigma_m$. The cases $(N, m) = (4, 2)$ or $(6, 3)$ are easy to verify. Consequently:

**Lemma 5.** With the dimension restriction of Theorem 1, all automorphisms of $H^*(G_m(C^N); \mathbb{Z})$ are induced by maps which are equivariant between $G_m(C^N)$ with action $\alpha$ and $G_m(C^N)$ with action either $\alpha$ or $\bar{\alpha}$.

In proving Theorem 1, we will also establish homotopy rigidity for any value of $N = 2m$ where Lemma 5 remains valid.

It is also helpful to define a filtration of $R[ h_1, \ldots, h_m]$. Let $J_k$ be the ideal
generated by the elements of $R$ having degree greater than or equal to $k$. (Here the degree of an element is one half its dimension as a cohomology class; i.e. \( \deg x_i = 1 \).) Clearly \( J_r \cdot J_r = J_{r+s} \) and \( \cap_{k=0}^\infty J_k = \{0\} \). Also define $S$ to be the linear span (over $R$) of all singletons $h_k$. Notice that the generators of the ideal of relations in $H_k^* \otimes \mathbb{Z}$ are elements of $S$. The following is the essential technical lemma required to prove Theorem 1. We will use the convention $h_x = 0$ if $x < 0$.

**Lemma 6.** Suppose $\Phi$ is a graded endomorphism of the algebra $R[h_1, \ldots, h_m]$ satisfying

\[
\Phi(\sigma_i) = \sigma_i + A_i \quad \text{modulo } J_{k+1}, \quad (k > 1)
\]

where $A_i \in J_k$ for $1 \leq i \leq m$.

Define elements $P_{rj}$ by

\[
P_{rj} = \left[ \sum_{i=1}^m (-1)^{i-1} A_i h_{j-i} \right] h_{r-j}.
\]

Then

\[
\Phi(h_r) = h_r + \sum_{j=1}^r P_{rj} \quad \text{modulo } J_{k+1}.
\]

**Proof.** The proof is by induction on $r$. The cases $r < 1$ are trivial. Assume the truth of the lemma for $r < s$. Since $h_s = \sum_{i=1}^m (-1)^{i-1} \sigma_i h_{s-i}$, we note that the $P_{sj}$ satisfy the recurrence relation

\[
P_{sj} = \sum_{i=1}^m (-1)^{i-1} \sigma_i P_{s-i,j} \quad \text{for } s > j
\]

with the convention $P_{sj} = 0$ if $s < j$. Now

\[
\Phi(h_s) = \sum_{i=1}^m (-1)^{i-1} \Phi(\sigma_i) \Phi(h_{s-i})
\]

\[
= \sum_{i=1}^m (-1)^{i-1} (\sigma_i + A_i) \left( h_{s-i} + \sum_{j=1}^{s-i} P_{s-i,j} \right) \quad \text{modulo } J_{k+1}
\]

\[
= \sum_{i=1}^m (-1)^{i-1} \sigma_i h_{s-i} + \sum_{i=1}^m (-1)^{s-i} A_i h_{s-i}
\]

\[
+ \sum_{i=1}^m \sum_{j=1}^{s-i} (-1)^{j-1} \sigma_i P_{s-i,j} \quad \text{modulo } J_{k+1}.
\]

But

\[
\sum_{i=1}^m \sum_{j=1}^{s-i} (-1)^{j-1} \sigma_i P_{s-i,j} = \sum_{j=1}^{s-1} \sum_{i=1}^m (-1)^{j-1} \sigma_i P_{s-i,j} = \sum_{j=1}^{s-1} P_{sj}.
\]

Thus $\Phi(h_s) = h_s + \sum_{j=1}^s P_{sj}$. \( \square \)

One of the complications of using Theorem 2 is that the product of two basis elements of $H^*(G_m(C^N); \mathbb{Z})$ cannot in general be readily expanded in terms of the basis. Fortunately, if we express $A_i$ in terms of this basis, then the expressions $P_{rj}$ in Lemma 6 will contain at most $m + 1$-fold products of the $h_n$'s. An $m + 1$-fold
product will occur only when \( k = 1 \) and \( A_m \) has a term \( \omega h_1^{m-1} \) in its expansion. In all other cases, the monomials in \( P_{r/j} \) will already be basis elements.

To avoid computation of \( h_1^{m-1} h_j^{m-1} h_{i-j}^{m-1} \), Lemma 6 will be used only in the following case.

**Lemma 7.** In addition to the assumptions of Lemma 6, assume \( A_m \) has no \( \omega h_1^{m-1} \) term in its expansion. (This is automatic if \( k > 1 \).) Then if \( N > 2m \) and \( m > 2 \),

\[ \Phi(h_{N-m+1}) \in S \text{ (the singletons) modulo } J_{k+1} \]

if and only if \( A_i \in J_{k+1} \) for all \( i \).

**Proof.** By Lemma 6, \( \Phi(h_{N-m+1}) \in S \) iff

\[ \sum_{i=1}^{m} (-1)^{i-1} A_i h_{N-m+1} + j \text{ modulo } J_{k+1} \]

lies in \( S \). Since \( A_i \in J_k \), \( A_i = 0 \) for \( i < k \). Upon expressing \( A_i \) relative to the basis of Theorem 2, noting that all products are less than or equal to \( m \)-fold, and observing \( \sum_{j=1}^{N-m+1} h_{N-m+1-j} = h_{N-m+1} + \text{doubletons} \), it now follows by induction on \( i \) that \( A_i \in J_{k+1} \) for all \( i \).

**Proof of Theorem 1.** We shall assume \( N > 2m \) and \( m > 2 \). (The case \( m = 1 \) is complex projective space where the technique of this paper works almost trivially upon factoring the relation \( h_1 \) into linear factors.)

By Theorem 3, the isomorphism \( f^* \) on equivariant cohomology is induced by an endomorphism \( \Phi \) of \( R[h_1, \ldots, h_m] \). Because the relations in \( H^*_F \) occur in dimensions above those of the generators, \( \Phi \) is uniquely determined by \( f^* \). Modulo \( J_1 \), \( \Phi \) must give one of the automorphisms of \( Z[h_1, \ldots, h_m] \) described in Lemma 5. As each is induced by an equivariant map, by composing with one of these, we may assume

\[ \Phi(\sigma_i) = \sigma_i + A_i \]

where \( A_i \in J_1 \) for \( 1 < i < m \).

Suppose \( A_m \) has a term \( \omega h_1^{m-1} \) in its expansion where \( \omega \in R \). Let \( \chi \) be the character with weight \(-w\). Then replacing \( \alpha \) by \( \alpha \otimes \chi \) replaces \( \sigma_m \) by the \( m \)-th elementary symmetric function of \( t_1 - w, t_2 - w, \ldots, t_m - w \). Modulo \( J_1 \), this is just \( \sigma_m - \omega \sigma_{m-1} \). However it follows readily by induction that the expansion of \( \sigma_{m-1} \) in terms of the basis of Theorem 2 contains \( h_1^{m-1} \) with coefficient 1. Thus with the replacement of \( \alpha \) by \( \alpha \otimes \chi \), we may assume that \( A_m \) has no \( \omega h_1^{m-1} \) terms in it.

Now \( \Phi \) must take the ideal \( \langle f_{N-m+1}^a, \ldots, f_N^a \rangle \) into the ideal \( \langle f_{N-m+1}^b, \ldots, f_N^b \rangle \). In particular, \( \Phi(f_{N-m+1}^a) = f_{N-m+1}^b \). Notice that both \( f_{N-m+1}^a \) and \( f_{N-m+1}^b \) belong to \( S \). But modulo \( S \) and \( J_2 \), \( \Phi(f_{N-m+1}^a) = \Phi(h_{N-m+1}) \). Hence by Lemma 7, \( A_i \in J_2 \). Continuing shows \( A_i \in J_k \) for all \( k \); i.e. \( A_i = 0 \) for all \( i \). Thus \( \Phi \) is the identity.

As \( \Phi \) preserves the ideal of relations, \( \Phi(f_{N}^a) = \sum_{i=0}^{m-1} u_i f_{N-i}^b \). Consideration of this ideal modulo \( J_1 \) shows \( u_0 = 1 \) and \( u_i \in J_1 \) for \( i > 0 \). Inductively considering this relation modulo \( J_k \) for \( k = 1, 2, 3, \ldots, m \) shows in fact that \( u_i = 0 \) for \( i > 0 \). Thus \( \Phi(f_{N}^a) = f_{N}^b \). But the coefficients of \( h_k \) in these relations are up to sign just the
$N - k$th symmetric functions of the weights of the two representations. Accordingly, all symmetric functions of the weights of $\alpha$ and $\beta$ must agree. Since the weights themselves are just the roots of the polynomial $h(t) = \sum_{j=0}^{N} (-1)^j \rho_j t^{N-j}$, it is immediate that $\alpha$ and $\beta$ have the same set of weights; i.e. that they are equivalent as linear representations.

Notice with trivial changes in dimensions that the analogous result for symplectic linear actions on quaternionic Grassmann manifolds also follows.

**Bibliography**


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