SPECIAL HANDLEBODY DECOMPOSITIONS OF SIMPLY-CONNECTED ALGEBRAIC SURFACES

RICHARD MANDELBAUM

Abstract. In this article we prove that any nonsingular complete-intersection surface admits a handlebody decomposition with no 1- and 3-handles. This generalizes results of Rudolph, Harer and Akbuluf on hypersurfaces of $\mathbf{CP}^3$.

Introduction. Among the problems posed at the Stanford Conference (24th Summer Research Institute, August, 1976) is the following [K, Preliminary List].

Problem 50 (Kirby). Does every simply-connected closed 4-manifold have a handlebody decomposition without 1-handles? Without 1- and 3-handles?

Rudolph [R] shows that any nonsingular hypersurface of $\mathbf{CP}^3$ has a handlebody decomposition without 1-handles (or dually, without 3-handles), however, Rudolph's method does not allow one to eliminate both 1- and 3-handles simultaneously. In December, 1976, at a lecture at the Institute for Advanced Study, Kirby exhibited a handlebody decomposition of the Kummer surface without both 1- and 3-handles. See [HKK].

The Kummer surface is diffeomorphic to a nonsingular quartic in $\mathbf{CP}^3$ and we sketch here the following generalization of the [HKK] result exhibited by Kirby.

Theorem. Suppose $X$ is a nonsingular complete intersection of $k$ distinct hypersurfaces $V_1, \ldots, V_k$ in $\mathbf{CP}^{k+2}$. Then $X$ has a handlebody decomposition without 1- and 3-handles.

1. Lefschetz fibrations.

Definition. Let $V$ be an algebraic surface and suppose $L$ is a pencil of curves on $V$. Then we shall say $L$ is a Lefschetz pencil if and only if

1. the generic element of $L$ is nonsingular and irreducible.

2. $L$ has only a finite number of singular elements, each of which has only one ordinary double point as its singularity.

We recall [W], [Z] that every algebraic surface admits Lefschetz pencils. (For more details on Lefschetz pencils see especially [AF], [W].)

Furthermore the Lefschetz pencil $L$ gives rise to a rational map $f$ of $V$ to $\mathbf{CP}^1$. If this map is a morphism $f: V \to \mathbf{CP}^1$ we shall call $L$ a Lefschetz fibration. It is clear that every Lefschetz pencil on $V$ gives rise to a Lefschetz fibration $\tilde{f}: \tilde{V} \to \mathbf{CP}^1$ of $\tilde{V} = \{ V \text{ blown up at the base points of } L \}$ onto $\mathbf{CP}^1$. (See [AF].)
Now suppose $f: V \to \mathbb{C}P^1$ is a Lefschetz fibration and suppose $a_1, \ldots, a_n \in \mathbb{C}P^1 - \{0, \infty\}$ are its critical values. Then for an appropriate choice of paths $\lambda_i$ in $\mathbb{C}P^1$ connecting $0$ to $a_i$ we can define as in [Z, p. 135], [MM1] Lefschetz vanishing cycles $\delta_i$ on $V_0 = f^{-1}(0)$ and Lefschetz relative cycles $\Delta_i \subset f^{-1}(\lambda_i) \subset V$.

We have the following theorem [AF] relating the topology of $V$ and $V_0$.

**Theorem 1.** Let $V_\infty = f^{-1}(\infty)$. Then $V - V_\infty$ has the homotopy type of $V_0$ with $n$ 2-discs $\Delta_i$ attached along the $\delta_i$.

In fact it is easy to see that we have a handlebody description of $V$ by means of its decomposition into $V_0$, the $\Delta_i$, $V_\infty$.

We now formulate:

**Definition.** Let $f: V \to \mathbb{C}P^1$ be a Lefschetz fibration and suppose that the generic fiber of $f$ has genus $g$. Suppose further there exist a system of paths in $\mathbb{C}P^1$ such that we can define Lefschetz vanishing and relative cycles $\delta_i$ on $V_0$, $\Delta_i$ in $V$ with the following property.

**Property W.** There exists a subsystem $\delta_1, \ldots, \delta_{2g}$ of Lefschetz vanishing cycles such that $B = \bigcup_i^{2g} \delta_i$ is a bouquet of $2g$ circles with $V_0 - B$ homeomorphic to a 2-disc.

Then we shall call $f: V \to \mathbb{C}P^1$ an exceptional Lefschetz fibration. We note the following result from [Wb].

**Theorem 2.** Let $n_0, n_1, \ldots, n_k$ be positive integers. There exists an algebraic curve $V_0 \subset \mathbb{C}P^2$ of degree $n = \prod_i^{k} n_i$ such that

1. $V_0$ has $n_0 \cdot n_1 \cdot \ldots \cdot n_i \cdot (n_i + 1)$ ordinary singular points of order $n_{i+1} \cdot n_{i+2}$ for $i = 0, 1, \ldots, k - 1$ and no other singular points.

2. There exist closed discs $D_1, D_2, \ldots, D_{2g}$ in $\mathbb{C}P^2$ (where $g$ denotes the genus of $V_0$), such that $D_i \cap V_0 = \partial D_i$ and $D_i \cap D_j = P$ for $i \neq j$ (where $P$ is a fixed point on $V_0$), and $D_i$ does not meet singular points of $V_0$, $i = 1, 2, \ldots, 2g$.

3. Let $\mathbb{C}P^2$ be $\mathbb{C}P^2$ blown up by $\sigma$-processes at the singular points of $V_0$. Let $\tilde{A}$ denote the proper image of $A$ on $\mathbb{C}P^2$ for any $A \subset \mathbb{C}P^2$. Then the bouquet of 1-spheres $\tilde{M} = \bigcup_i^{2g} \tilde{\partial}D_i$ generates the fundamental group $\pi_1(\tilde{V}_0, \tilde{P})$ and its complement $\tilde{V}_0 - \tilde{M}$ is homeomorphic to an open disc.

We note that the proof of Theorem 2 actually implies

**Proposition 3.** Let $V$ be a nonsingular complete intersection surface in some projective space $\mathbb{P}^N$. Let $H_V$ be a hyperplane section of $V$ and suppose $H_V^2 = m$. Then for every integer $n$ there exists an exceptional Lefschetz fibration $f_n: V_n \to \mathbb{C}P^1$ of $V'_n = V$ blown up at $mn^2$ points, with fiber the proper transform of a hypersurface section of $V$ of degree $n$.

2. Handlebody decompositions.

**Definition.** Suppose $f_1: V_1 \to S^2$, $f_2: V_2 \to S^2$ are Lefschetz fibrations of curves of genus $g$.

Let $a_1, a_2$ be regular values of $f_1, f_2$ resp. and let $D_1, D_2$ be discs around $a_1, a_2$ not containing any critical values.
Let $h: \partial D_1 \to \partial D_2$ be an orientation reversing diffeomorphism and identifying $f_i^{-1}(D_i)$ with $D_i \times F$ for $i = 1, 2$ (where $F$ is a nonsingular curve of genus $g$) extend $h$ to $g: f_1^{-1}(D_2) \to f_2^{-1}(D_2)$ by the identity on the second factor.

Let

$$V = V_1 - f_1^{-1}(D_1) \cup S, \quad V_2 - f_2^{-1}(D_2), \quad S = S^2 - D_1 \cup_h S^2 - D_2$$

and define $f: V \to S$ by

$$f[V_i - f_i^{-1}(D_i)] = f_i[V_i - f_i^{-1}(D_i)], \quad i = 1, 2.$$

It is clear that $S \approx S^2$ and $f: V \to S$ is a Lefschetz fibration of curves of genus $g$ (provided $V$ is algebraic. In all our applications we shall know a priori that $V$ is algebraic.) We write $V$ as $V = V_1 \oplus V_2$ and call it the direct Lefschetz sum of $V_1$ and $V_2$. We then have

**Proposition 4.** Let $f_i: V_i \to S^2$ be exceptional Lefschetz fibrations of curves of the same genus $g$ for $i = 1, 2$.

Let $V = V_1 \oplus V_2$ and $f: V \to S^2$ be defined as above. Then $V$ admits a handlebody decomposition with no 1- or 3-handles.

**Proof.** Since the $f_i$ are exceptional Lefschetz fibrations of genus $g$ there exist systems of paths in $\mathbb{C}P^1$ such that we can define Lefschetz vanishing and relative cycles $s^j_i$ on $V_{i,0}$ and $\Delta'_j$ on $V_i, i = 1, j = 1, \ldots, N_1; i = 2, j = 1, \ldots, N_2$, such that there exist subsystems $s^j_i, i = 1, 2; j = 1, \ldots, 2g$, of the Lefschetz vanishing cycles satisfying Property W.

Let $V'_i = V_{i,0} \cup (\cup_{j=1}^{2g} \Delta'_j), i = 1, 2$. Then $V'_i = V_i - T(V_{i,\infty})$ admits the following decomposition.

$$V'_i \approx N(V'_{i,0}) \cup \bigcup_{j=1}^{N_i} N(\Delta'_j) = N(V'_i) \cup \bigcup_{j=g+1}^{N_i} N(\Delta'_j)$$

where $\approx$ means “deformation retract”, $N(A)$ is a regular nbhd of $A \subset V'_i$ in $V'_i$ and $T(X)$ is a tubular neighborhood of $X$.

Then by Property W we deduce that $N(V'_i)$ is homeomorphic to $D^4 \cup \{2\text{-handle}\}$. In particular let $p_i \in V'_{i,0} - \bigcup_{j=1}^{2g} \Delta'_j$. Then there exists a 2-disc $D_i \supset p_i$ in $V_{i,0}$ and we have that $H'_0 = N((V'_{i,0} - D_i) \cup (\cup_{j=1}^{2g} \Delta'_j))$ is a 0-handle while $N(D_i), N(\Delta'_j), i = 1; j = 2g + 1, \ldots, N_1, i = 2; j = 2g + 1, \ldots, N_2$, are 2-handles attached to $H'_0$ along $N(D_i) \cap \partial H'_0, N(\Delta'_j) \cap \partial H'_0$ respectively.

Thus $V'_i$ admits a handlebody decomposition consisting only of a 0-handle and $N_1 - 2g + 1$ 2-handles.

In particular then by duality we obtain that since $V = V'_1 \cup V'_2$ then $V$ admits a handlebody decomposition consisting only of a 0-handle, $N_1 + N_2 + 2 - 4g$ 2-handles and a 4-handle as desired.

We now have

**Theorem 5.** Suppose $X$ is a nonsingular complete intersection of $k$ distinct hypersurfaces $V_1, \ldots, V_k$ in $\mathbb{C}P^k + 2$. Then $X$ has a handlebody decomposition without 1- and 3-handles.
PROOF. Since \( X \) is a complete intersection we may, using Corollary 6.2 of \([\text{MM}2]\), assume that there exist \( k \) hypersurfaces \( V_i(n_i) \) of degree \( n_i \) in \( \mathbb{CP}^{k+2} \) such that setting \( Y = \bigcap_{i=1}^{k} V_i(n_i) \) we have that \( Y \) is nonsingular and \( Y \) intersects \( V_k(n_k) \) transversely. Furthermore, as our theorem is obvious for \( \mathbb{CP}^2 \) we may also assume without loss of generality that \( n_k > 2 \). Using Corollary 6.2 of \([\text{MM}2]\) we see that there exists a hypersurface \( V'_k \) of degree \( n_k - 1 \) in \( \mathbb{CP}^{k+2} \) and a hyperplane \( H \) in \( \mathbb{CP}^{k+2} \) such that \( V'_k, H \) intersect \( Y \) transversely and \( V'_k, H, V_k \) have normal crossing in \( \mathbb{CP}^{k+2} \).

Let \( X' = Y \cap V'_k, X'' = Y \cap H, C = Y \cap V'_k \cap H, m = X \cdot V'_k \cdot H, m_1 = X' \cdot H^2, m_2 = (X'')^2 \cdot H \) (where \( \cdot \) is multiplication of cycles in the homology ring of \( \mathbb{CP}^{k+2} \) and we identify \( H_0(\mathbb{CP}^{k+2}) \) with \( \mathbb{Z} \)).

Then again using Corollary 6.2 of \([\text{MM}2]\), we obtain \( X = \sigma_m(X') - T(C) \cup X'' - T(C) \) where \( \sigma_m(X') \) is \( X' \) blown up at \( m \) points and \( T(C), T(C) \) are tubular neighborhoods of \( C = (\text{strict image of } C \text{ in } \sigma_m(X')) \), and \( C \) respectively. But using \([P, \text{Chapter 2, Corollary 3}]\), we can obtain \( X = \sigma_m(X') - T(C') \cup \sigma_m(X'') - T(C'') \) where \( C', C'' \) denote strict images.

But using Proposition 3 we see that \( \sigma_m(X') \) and \( \sigma_m(X'') \) both admit exceptional Lefschetz fibrations with \( C' \), resp. \( C'' \) a typical fiber. Thus we obtain that \( X = \sigma_m(X') \oplus \sigma_m(X'') \) and so by Proposition 4 we have that \( X \) admits the requisite handlebody techniques.

Using similar techniques together with the methods of \([M]\) we can also obtain

**Theorem 6.** Suppose \( X \) is either a nonsingular simply-connected elliptic surface with no more than one multiple fiber or a cyclic branched cover of a nonsingular complete intersection surface with branch locus a nonsingular complete intersection curve. Then \( X \) admits a handlebody decomposition without 1- and 3-handles.

**References**


**Department of Mathematics, University of Rochester, Rochester, New York 14627**