A TOPOLOGICAL CHARACTERIZATION OF A CLASS OF CARDINALS

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ABSTRACT. Let $m$ be the first measurable cardinal. We say that a cardinal $\alpha$ is Ulam-stable if, on the discrete space $D(\alpha)$ of cardinal $\alpha$, every filter with $m$-intersection property can be extended to an ultrafilter with $m$-intersection property. The main result we prove is the following: $\alpha$ is Ulam-stable if and only if its Hewitt-Nachbin realcompletion $\nu D(\alpha)$ is paracompact.

0. Introduction. Until now, various classes of cardinals have been considered in mathematical literature (see [1], [6]); a certain amount of attention has been devoted to characterize, in topological terms, some set-theoretic relations connected with them.

Keisler and Tarski [6] introduced a binary relation $\mathcal{R}$, in the class of all cardinals, which Comfort-Negrepontis in their paper [1] modified as follows: $\alpha \nRightarrow \beta$ provided that there is, on the discrete space of cardinal $\beta$, an $\alpha$-complete filter that cannot extend to an $\alpha$-complete ultrafilter. In the paper referred to, such set-theoretic relation is described from a topological point of view. The aim of the present paper goes along these lines: we find a topological characterization for the class of all cardinals which do not satisfy the above relation for fixed $\alpha = m$ (first Ulam-measurable cardinal). We call them Ulam-stable cardinals and denote their class by $\mathcal{S}_m$. It is well known that if $\alpha$ is a nonmeasurable cardinal (in the sense of [5]) then the discrete space $D(\alpha)$ is realcompact, and thus it coincides with its Hewitt-Nachbin realcompletion $\nu D(\alpha)$, which is obviously paracompact.

Assuming the existence of an Ulam-measurable cardinal, the problem we solve here is to determine the class of all cardinals $\xi$ for which $\nu D(\xi)$ is paracompact. We prove that $\nu D(\xi)$ is paracompact if and only if $\xi$ is Ulam-stable, i.e., $\xi$ belongs to $\mathcal{S}_m$.

1. Basic notions and results. We denote ordinal numbers by Greek letters; a cardinal number is an initial ordinal. We indicate by $v$ the first infinite cardinal and by $\alpha +$ the smallest cardinal greater than $\alpha$. A cardinal $\alpha$ is said to be regular if it cannot be expressed as the sum of fewer than $\alpha$ cardinals each smaller than $\alpha$. The discrete space of cardinal $\beta$ is denoted by $D(\beta)$. A filter $\mathcal{F}$ on $D(\beta)$ has the $\alpha$-intersection property (abbr. $\alpha$-i.p.) if $\bigcap \mathcal{B} \neq \emptyset$ whenever $\mathcal{B} \subseteq \mathcal{F}$ and $|\mathcal{B}| < \alpha$; moreover, if $\bigcap \mathcal{B} \in \mathcal{F}$ for $\mathcal{B} \subseteq \mathcal{F}$ and $|\mathcal{B}| < \alpha$, then $\mathcal{F}$ is said to be $\alpha$-complete. An ultrafilter with $\alpha$-i.p. is clearly $\alpha$-complete. We recall here that a cardinal $\alpha$ is

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(Ulam)-measurable if there exists a nonprincipal $v_+^+$-complete ultrafilter on $D(\alpha)$; let $m$ denote the first measurable cardinal. It is known that if $\alpha > m$ then $\alpha$ is a measurable cardinal.

Furthermore, $D(\alpha)$ is not realcompact and its Hewitt-Nachbin realcompletion $vD(\alpha)$ can be thought of as the space of all $m$-complete ultrafilters on $D(\alpha)$. Other equivalent definitions for measurable cardinals (involving measures) are to be found in [3] and [5]. We say that a cardinal $\alpha$ is $m$-stable (or Ulam-stable) if, on $D(\alpha)$, every filter with $m,i.p.$ extends to an $m$-complete ultrafilter. We say that $\alpha$ is strongly measurable if $\alpha$ is regular and every filter on $D(\alpha)$ having the $a,i.p.$ extends to an $\alpha$-complete ultrafilter. We observe that if $\alpha$ is measurable, every $m$-complete filter on $D(\alpha)$ extends to an $m$-complete ultrafilter if and only if every filter with $m,i.p.$ does.

Furthermore, we need some topological notions and results. Let $X$ be a Tychonoff space; $M$ a dense subset of $X$.

We denote by $\mathcal{T}_M$ the class of all mappings of $M$, taking values in a metric space, which can be continuously extended to $X$. If $\mathcal{F}$ is a filter on $M$, we denote by $\mathcal{F}$ the extension of $\mathcal{F}$ to $X$, namely the filter on $X$ generated by $\mathcal{F}$. We state here without proof two theorems.

**Theorem A (Corson [4]).** For a Tychonoff space $X$ the following are equivalent:

(I) $X$ is paracompact;

(II) If $\mathcal{F}$ is a filter on $X$ such that the image of $\mathcal{F}$ has a cluster point in any metric space into which $X$ is continuously mapped, then $\mathcal{F}$ has a cluster point in $X$.

The second result we quote here was obtained jointly by W. W. Comfort and Teklehaimanot Retta and communicated to me February 1, 1977.

**Theorem B (Comfort-Retta).** If $vD(m)$ is paracompact, then $m$ is strongly measurable.

**Corollary.** Suppose $\alpha > m$. If $vD(\alpha)$ is paracompact, then $m$ is strongly measurable.

2. **Lemmas.** We need two lemmas. The first is a strengthened version of Theorem A.

**Lemma 1.** Let $X$ be a Tychonoff space and $M$ a dense subset of $X$. Then the following are equivalent:

(I) $X$ is paracompact;

(II) If $\mathcal{F}$ is a filter on $M$ such that $f(\mathcal{F})$ has a cluster point for each $f \in \mathcal{T}_M$, then the extension $\mathcal{F}$ of $\mathcal{F}$ to $X$ has a cluster point in $X$.

**Proof.** (I) $\rightarrow$ (II) is obvious. For the converse, let $\mathcal{F}$ be a filter on $X$ such that the image of $\mathcal{F}$ has a cluster point in any metric space into which $X$ is continuously mapped: we must show that $\mathcal{F}$ clusters in $X$. Take the n.b.d. filter $\mathcal{U}_f(\mathcal{F})$ and its trace $\mathcal{G}$ on $M$. It can be easily checked that $f(\mathcal{G})$ has a cluster point for each $f \in \mathcal{T}_M$ and thus $\mathcal{F}$ clusters in $X$ by the hypothesis.
Consequently, \( F \) has a cluster point in \( X \) and so \( X \) is paracompact applying Theorem A.

**Lemma 2.** Let \( \alpha, \beta \) be infinite cardinals with \( \beta > \alpha \) and \( F \) a filter on \( D(\beta) \) which fails to have the \( \alpha \)i.p. Then there is a cardinal \( \gamma < \alpha \) and a family \( (A_\xi)_{\xi < \gamma}, A_\xi \in F \) with the following properties:

(a) \( \bigcap_{\xi < \gamma} A_\xi = \emptyset \).
(b) \( \bigcap_{\xi < \gamma} A_\xi \neq \emptyset \), and \( \bigcap_{\xi < \gamma} A_\xi \not
\in F, \forall \xi < \gamma \).

**Proof.** We divide the proof into several steps. Since \( F \) fails to have the \( \alpha \)i.p., there is a cardinal \( \theta < \alpha \) and a family \( (A_\eta)_{\eta < \theta} \) with \( \bigcap A_\eta = \emptyset \). Moreover we can suppose, without loss of generality, that \( \theta \) is the smallest cardinal having the above property, i.e. \( \bigcap_{\eta < \theta} A_\eta \neq \emptyset, \forall \theta' < \theta \).

(I) \( \forall \eta < \theta, \exists \eta < \theta, \bigcap_{\eta < \theta} A_\eta = A_\eta \).

For, take \( y \in \bigcap_{\eta < \theta} A_\eta \) and \( \lambda < \theta \) such that \( y \notin A_\lambda \). Then \( \bigcap_{\eta < \theta} A_\eta = A_\lambda \). If \( \lambda \) is the smallest ordinal \( \mu \) such that \( \bigcap_{\eta < \theta} A_\eta \subseteq A_\mu \), we have: \( \bigcap_{\eta < \theta} A_\eta \subseteq A_\eta \) and \( \bigcap_{\eta < \theta} A_\eta \subseteq \bigcap_{\eta < \theta} A_\eta \).

(II) \( \eta > \eta \). Obvious.

(III) \( \eta > \eta \). Let us suppose \( \eta > \eta \). Then, by (I) and (II) we have the following chain of inclusions: \( \bigcap_{\eta < \theta} A_\eta \subseteq \bigcap_{\eta < \theta} A_\eta \subseteq \bigcap_{\eta < \theta} A_\eta \subseteq A_\eta \) which is absurd.

Let us denote by \( K \) the set of all \( \eta \) for \( \eta < \theta \).

(IV) \( \bigcap_{\eta \in K} A_\eta = \emptyset \). For \( z \in \bigcap_{\eta \in K} A_\eta \), let us call \( \eta_0 \) the first index with \( z \notin A_{\eta_0} \). It is easy to verify that \( z \notin A_{\eta_0} \).

(V) \( \bigcap_{\eta < \theta} A_\eta \subseteq A_\eta \). This follows from (III) and from the fact that \( \bigcap_{\eta < \theta} A_\eta \subseteq \bigcap_{\eta < \theta} A_\eta \).

Now we set \( |K| = \gamma \). Of course \( \gamma < \theta < \alpha \), and indexing \( K \) by ordinals \( \xi < \gamma \) we obtain respectively (a) from (IV) and (b) from (V).

**3. Main Result.** We can now prove our main result. We have the following

**Theorem.** Let \( \beta \) be a cardinal number. \( \beta \) is Ulam-stable if and only if the space \( vD(\beta) \) is paracompact.

**Proof.** If \( \beta \) is nonmeasurable, the theorem is trivially true; hence we assume \( \beta > \mathfrak{m} \).

**Necessity.** Let us suppose \( \beta \) Ulam-stable, and show that \( vD(\beta) \) is paracompact. Let \( F \) be a filter on \( D(\beta) \) such that \( f(F) \) has a cluster point, \( \forall f \in F_{D(\beta)} \). It is enough to prove that \( F \) has the m.i.p. Then by the hypothesis \( F \) is contained in an ultrafilter \( \mathcal{A}_p, p \in vD(\beta) \), and consequently has a cluster point in \( vD(\beta) \), which turns out to be paracompact for Lemma 1. Suppose on the contrary that \( F \) has not the m.i.p. Then, by Lemma 2, there is a nonmeasurable cardinal \( \gamma \) and a family \( (A_\xi)_{\xi < \gamma}, A_\xi \in F \), for which the properties (a), (b) of the quoted lemma hold.

For every \( \xi < \gamma \), let \( x_\xi \) be a point such that \( x_\xi \in \bigcap_{\xi < \gamma} A_\xi \) and \( x_\xi \notin \bigcap_{\xi < \gamma} A_\xi \). Let us pose \( D = (x_\xi)_{\xi < \gamma} \). Now we are going to define a map \( \lambda: D(\beta) \to D \) in the following way: \( \lambda(x) = x_\xi \), where \( \xi \) is the first ordinal with \( x \notin A_\xi \).

Since \( D \) is a discrete space of nonmeasurable cardinality, it is realcompact. This implies that \( \lambda \) is continuously extendable to \( vD(\beta) \), namely \( \lambda \in F_{D(\beta)} \).
On the other hand, the filter \( \lambda(\mathcal{F}) \) does not have cluster points in \( D \), because the family \( (\lambda(A_\mu) \subseteq \gamma) \) is free. For it, if \( z \in \cap \lambda < \gamma \lambda(A_\mu) \) then \( z = x_\mu \) for a certain index \( \mu < \gamma \) and moreover there is a point \( x \in A_\mu \) with \( z = \lambda(x) \). But by the definition of \( \lambda, \lambda(x) = x_\mu \) where \( x \not\in A_\mu \), so \( x_\mu \) cannot belong to \( \lambda(A_\mu) \). This is absurd, since we had supposed that \( f(\mathcal{F}) \) had a cluster point, for each \( f \in \mathcal{F}_D(\beta) \); consequently \( \mathcal{F} \) must have the m.i.p., and the proof is completed.

Sufficiency. Let us suppose \( vD(\beta) \) paracompact, and let \( \mathcal{F} \) be a filter on \( D(\beta) \) with m.i.p. We must show that \( \mathcal{F} \) is contained in an ultrafilter \( A_p \), for \( p \in vD(\beta) \).

By Lemma 1, it is enough to prove that \( f(\mathcal{F}) \) clusters, for each \( f \in \mathcal{F}_D(\beta) \). Let us point out firstly that \( f(D(\beta)) \) must be a realcompact metric space \( \forall f \in \mathcal{F}_D(\beta) \).

Indeed, if \( f(D(\beta)) \) contained a discrete, closed and measurable set, \( f \) could not be extended to \( vD(\beta) \). The proof is easy and we omit it.

Suppose now that there is a \( g \in \mathcal{F}_D(\beta) \) such that \( g(\mathcal{F}) \) does not cluster in a metric, realcompact space \( D \). Since \( g(\mathcal{F}) \) has not cluster points there exists a locally finite open cover \( (U_i)_{i<\sigma}, U_i \subseteq D \), and a family \( (F_i)_{i<\sigma}, F_i \in \mathcal{F}, \) with \( \cap U_i \cap g(F_i) = \emptyset \).

Clearly \( \sigma \) is not measurable, because if it were, one could select a discrete, closed, measurable set contrary to the realcompactness of \( D \). We have \( \cap (D - U_i) = \emptyset \) with \( (D - U_i) \in g(\mathcal{F}) \), against the fact that \( \mathcal{F} \) and consequently \( g(\mathcal{F}) \) have the m.i.p. This completes the proof of our theorem.

REFERENCES


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